INVOLUTIONAL TRANSFORMATIONS AND THE METHOD OF SUCCESSIVE APPROXIMATIONS FOR ELLIPTIC EQUATIONS

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Abstract

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INVOLUTIONAL TRANSFORMATIONS AND THE METHOD OF SUCCESSIVE APPROXIMATIONS FOR ELLIPTIC EQUATIONS

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Consider, inside a bounded domain Ω of n-dimensional Euclidean space, the quasilinear equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} [a_i(x, u, p_j)] - a_0(x, u, p_j) = 0, \tag{1}$$

where the functions a_i $(i = 0, 1, \dots, n)$ depend on $x \in \Omega$, the unknown function u, and its first derivatives

$$p_j = \partial u / \partial x_j \quad (j = 1, \dots, n).$$

With respect to the functions a_i $(i = 0, 1, \dots, n)$, we shall assume that the following conditions are satisfied:

1) For all x, u, p_j and arbitrary real $\xi_0, \xi_1, \dots, \xi_n$, the inequality

$$\sum_{i,k=0}^{n} \frac{\partial a_i}{\partial p_k} \xi_i \xi_k \geqslant f(T) \sum_{i=0}^{n} \xi_i^2, \tag{2}$$

holds, where $u = p_0$, $T^2 = \sum_{i=0}^{n} p_i^2$, and $f(T)$ is a continuous, strictly positive function for $T \geqslant 0$, having, as $T \to +\infty$, order $O(\ln^\gamma T)$ $(0 < \gamma < 1)$.

2) For all x, u, p_j, the inequalities

$$\left| \frac{\partial a_i}{\partial p_k} \right| < A f(T), \tag{3}$$

$$|a_i| < B f(T) T, \tag{4}$$

$$\sum_{i=0}^{n} a_i^2 \geqslant C \sum_{i=0}^{n} p_i^2 (1 + \ln^{2\gamma} T), \tag{5}$$

hold, where A, B, and C are certain positive constants.

3) For any function u continuous together with its first partial derivatives, the functions a_i $(i = 0, 1, \dots, n)$ and their derivatives with respect to all arguments are functions summable over the domain Ω with exponent $p > n$.

We shall seek a generalized solution of the problem for equation (1) satisfying, on the sufficiently smooth boundary Γ of the domain Ω, the boundary condition

$$u|_{\Gamma} = 0. \tag{6}$$

By a generalized solution of problem (1)–(6) we mean such a func-

function $u \in \overset{\circ}{W}{}_2^{(1)}$, for which the integral

$$\int_\Omega T^2 \left(1 + \ln^{2\gamma} T\right) \, dx \qquad (0 < \gamma < 1) \tag{7}$$

is finite, and for every $v \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ the relation

$$\int_\Omega \left[a_i(x, u, p_j) \frac{\partial v}{\partial x_i} + a_0(x, u, p_j) v \right] dx = 0. \tag{8}$$

We shall now transform problem (1)–(6).

Equality (3) will be satisfied if there is found a vector-function $\vec{\lambda}\{\lambda_1, \dots, \lambda_n\}$, possessing the generalized divergence

$$\operatorname{div} \vec{\lambda} = \sum_{i=1}^{n} \frac{\partial \lambda_i}{\partial x_i},$$

and satisfying the system of equations

$$a_i(x, u, p_j) = \lambda_i \quad (i = 1, 2, \dots, n); \qquad a_0(x, u, p_j) = \operatorname{div} \vec{\lambda}. \tag{9}$$

Since the Jacobian of system (9), by inequality (2), is different from zero, this system admits a unique solution

$$p_i = p_i(x, \lambda_j, \operatorname{div} \vec{\lambda}) \qquad (i = 0, 1, \dots, n).$$

In order that the functions $p_i(x, \lambda_j, \operatorname{div}\vec{\lambda})$ lead to a generalized solution of our problem, it is required that, for every sufficiently smooth vector function $\mu\{\mu_1, \dots, \mu_n\}$ vanishing in the boundary strip, the equality

$$\sum_{i=1}^{n} \int_{\Omega} \left(\frac{\partial p_0}{\partial x_i} - p_i \right) \mu_i \, dx = 0$$

be fulfilled.

Integrating by parts, we arrive at the equality

$$\int_{\Omega} \left(p_0 \operatorname{div} \vec{\mu} + \sum_{i=1}^{n} p_i \mu_i \right) dx = 0, \tag{10}$$

from which there must be found such a vector-function $\vec{\lambda}$ that belongs to $\mathcal{L}_2(\Omega)$ and on the boundary Γ satisfies the condition

$$P_0\big|_{\Gamma} = 0.$$

Suppose that from the equality $a_0(x, u, p_j) = 0$ it follows that $u = 0$. Then the preceding boundary condition may be replaced by the condition

$$\operatorname{div} \vec{\lambda}\big|_{\Gamma} = 0. \tag{11}$$

We shall show that problem (10)–(11) will have a solution to which the process of successive approximations converges.

In ([1]) we showed that, in the case when $a_i(x, u, p_j)$ satisfy the so-called condition of restricted nonlinearity, the solution of problem (1)–(6) can be found by a process of successive approximations, which converges in the metric $W_2^{(1)}$. At the same time, if the conditions of restricted nonlinearity are not fulfilled, then, as proved in our note ([1]), this process may diverge, despite the existence of a solution and its uniqueness.

It turns out that the application of the Friedrichs involution transformation (9) ([2]) to the boundary-value problem (1)–(2) leads to problem (10)–(11), for which the process of successive approximations may prove convergent. In the theory of plasticity this transformation transforms a problem expressed in terms of stress functions into an analogous problem expressed in terms of strain functions.

For the solution of problem (10)–(11) let us consider the successive approximations
$\vec{\lambda}^{(k)}\{\lambda_1^{(k)}, \dots, \lambda_n^{(k)}\}$ $(k = 0, 1, \dots)$, defined by the following scheme:

$$\int\limits_\Omega \left(\operatorname{div}\vec{\lambda}^{(k+1)}\operatorname{div}\mu + \sum_{i=1}^{n}\lambda_i^{(k+1)}\mu_i\right)\,dx = \int\limits_\Omega \left(\operatorname{div}\vec{\lambda}^{(k)}\operatorname{div}\mu + \sum_{i=1}^{n}\lambda_i^{(k)}\mu_i\right)\,dx - \alpha\int\limits_\Omega \Big[p_0\left(x, \operatorname{div}\vec{\lambda}^{(k)}, \lambda_j^{(k)}\right)\operatorname{div}\mu \tag{12}$$

$$\operatorname{div}\vec{\lambda}^{(k+1)}\big|_\Gamma = 0, \tag{13}$$

where as $\vec{\lambda}^{(0)}$ one may take any vector function from $W_p^{(2)}$ $(p > n)$, and α is some sufficiently small positive constant. Denote
$\lambda_0^{(k)} = \operatorname{div}\vec{\lambda}^{(k)}$ $(k = 0, 1, ...)$.

It can be proved, using equalities (12), that the successive approximations $\vec{\lambda}^{(k)}$ satisfy the inequality

$$\sum_{i=0}^{n}\int\limits_\Omega \left|\lambda_i^{(k+1)} - \lambda_i^{(k)}\right|^2 dx \le \left[1 - \varphi\left(\Lambda^{(k)}\right)\right]\sum_{i=0}^{n}\int\limits_\Omega \left|\lambda_i^{(k)} - \lambda_i^{(k-1)}\right|^2 dx, \tag{14}$$

where

$$\Lambda^{(k)} = \max_j\left(\left\|\lambda_j^{(k)}\right\|_C, \left\|\lambda_j^{(k-1)}\right\|_C\right)$$

and $\varphi(t)$ is a continuous function, nonnegative for $t \ge 0$, decreasing as $t \to +\infty$, and having, as $t \to +\infty$, order

$$O\left(\frac{1}{\ln^\gamma t}\right) \qquad (0 < \gamma < 1).$$

It can also be proved on the basis of (3) that the successive approximations $\vec{\lambda}^{(k)}$ will be continuous in Ω, together with their first derivatives, and for them the estimate

$$\left\|\vec{\lambda}^{(k)}\right\|_{C^{(1)}} \le (1+\beta)^k$$

holds, where β is some positive constant.

It then follows from inequality (14) that

$$\left\|\vec{\lambda}^{(k+1)} - \vec{\lambda}^{(k)}\right\|_{\mathcal{L}_2(\Omega)} \le \left(1 - \frac{\delta}{k^\gamma}\right)\left\|\vec{\lambda}^{(k)} - \vec{\lambda}^{(k-1)}\right\|_{\mathcal{L}_2(\Omega)},$$

where $\delta > 0$ is a sufficiently small constant. The last inequality guarantees convergence of the process; moreover it is easy to show that $\operatorname{div}\vec{\lambda}^{(k)}$ converges in the metric $\overset{\circ}{W}_2^{(1)}$, so that the boundary condition (13) is fulfilled in the sense of S. L. Sobolev. The order of convergence coincides with the order of convergence of the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^r},$$

where r is some positive constant greater than one. Thus, the following theorem is valid.

Theorem. *If for equation (1) all the conditions listed above are satisfied, then problem (1)–(6), as well as problem (10)–(11), have generalized solutions; moreover the solution of problem (10)–(11) can be obtained by the method of successive approximations according to scheme (12).*

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CITED LITERATURE

[1] A. I. Koshelev, DAN 142, No. 5 (1962). [2] K. Friedrichs, Nachr. Ges. Wiss. Göttingen, 13 (1929). [3] A. I. Koshelev, UMN, 13, issue 4 (82), 29 (1958).

Note: Figure translations are in progress. See original paper for figures.