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Abstract

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ADDITION THEOREMS FOR THE WEIGHT OF TOPOLOGICAL SPACES

(Presented by Academician P. S. Aleksandrov on 26 II 1963)

In the present note the following question* is considered:

Let a topological space X be the sum of a set of cardinality $\leq \tau$ of its subspaces, each of which has weight $\leq \tau$. Under what conditions can one assert that the space X also has weight $\leq \tau$?

A. Arkhangel' skii ([2]) proved that if a locally bicompact space Φ is the sum of a set of cardinality $\leq \tau$ of its subspaces, each of which has weight $\leq \tau$, or, more generally, if in a locally bicompact space Φ there exists a network** of cardinality $\leq \tau$, then Φ has weight $\leq \tau$. This theorem admits the following generalizations.

Theorem 1. *A necessary and sufficient condition for a completely regular space X, in which there exists a network of cardinality $\leq \tau$, to have weight $\leq \tau$ is the existence of a bicompactification $\widetilde{X}$ of the space X with remainder $\widetilde{X} \setminus X$ having weight $\leq \tau$ (or, more generally, having a network of cardinality $\leq \tau$).*

Proof. Necessity follows from Tikhonov' s theorem ([1]), sufficiency—from the cited theorem of A. Arkhangel' skii.

Obvious is

Lemma 1. *If a space X is the sum of a set of cardinality $\leq \tau$ of its open subsets X_t of weight $\leq \tau$, then the weight of the space X also does not exceed τ.*

Theorem 2. *Let a space X be the sum of a set of cardinality $\leq \tau$ of its subspaces X_t, the weight of each of which is $\leq \tau$. Then a necessary and sufficient condition for X to have weight $\leq \tau$ is that for every point $x \in X$ there exist a neighborhood Ox in the space X having weight $\leq \tau$.*

Proof. Necessity is obvious. We prove sufficiency.

Let B'_t be a base of cardinality $\leq \tau$ of open sets in the subspace X_t. Denote by B_t the collection of all sets G from B'_t for which there exists an open neighborhood O_G of the set G in the space X having weight $\leq \tau$. Then, if the condition of the theorem is fulfilled, B_t forms a base of open sets in the subspace X_t. Indeed, for any $x \in X_t$ there exists $G \in B'_t$ such that $x \in G \subseteq Ox$ (where Ox is a neighborhood of the point x in the space X whose weight is $\leq \tau$). And since for all $G' \in B'_t$ satisfying the condition $G' \subseteq G$ we have $G' \in B_t$, the system B_t contains a base of open neighborhoods in X_t of any point $x \in X_t$. Hence B_t is a base of the space X_t, and the cardinality of $B_t \leq \tau$, since $B_t \subseteq B'_t$. Hence

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* A historical note is given in ([2]). We only note that in its simplest form this question was posed by P. S. Aleksandrov and P. S. Urysohn as early as the 1920s, and the first substantial advance in its solution was obtained by Yu. M. Smirnov in 1956.

** By a network of a space X is meant such a system $\Sigma = \{A_\alpha\}$ of sets of this space that for any point $x \in X$ and any of its neighborhoods Ox there is an $A_\alpha \in \Sigma$ such that $x \in A_\alpha \subseteq Ox$ (see ([2])).

it follows that

$$\bigcup_{G \in \bigcup_t B_t} O_G = X$$

(where by O_G we still denote some open set in X containing G and of weight $\leq \tau$); and since the number of all sets O_G is no greater than τ, because the number of the sets G themselves satisfying the condition $G \subset \bigcup_t B_t$ is no greater than τ and they all have weight $\leq \tau$, it follows, by Lemma 1, that the weight of the space is $\leq \tau$.

Remark. What was essential for us in the proof was that the system B_t covers X_t and has cardinality $\leq \tau$. But for this it is sufficient that X_t be compact, starting with cardinality τ. Then X is also compact, starting with cardinality τ. In view of this remark and Lemma 1, the following becomes obvious.

Theorem 2′. *In order that the weight of the space X not exceed τ, it is necessary and sufficient that for every point $x \in X$ there exist a neighborhood Ox having weight $\leq \tau$, and that the space X be compact, starting with cardinality τ.*

Definition. A subset A of a space X will be called **τ-dense in itself** if for every point $x \in A$ and every neighborhood Ox of the point x, the set $A \cap Ox$ has cardinality $\geq \tau$.

For the space X, whose weight we shall suppose to be $> \tau$, denote by W the set of all $x \in X$ for which every neighborhood Ox has weight $> \tau$, and by τ_1 the least cardinal number $> \tau$. Then:

Theorem 3. *Let the space X have weight $> \tau$, let every point $x \in X$ have a base of neighborhoods of cardinality $\leq \tau$, and let, finally, X be the sum of a set of cardinality $\leq \tau$ of its subspaces X_t, each of which has weight $\leq \tau$. Then W is a τ_1-dense-in-itself set of cardinality $\geq \tau_1$, and the set $X \setminus W$ has weight $\leq \tau$.*

Proof. For every point $x \in X \setminus W$ there exists a neighborhood Ox in the space X whose weight is $\leq \tau$. Obviously, for such a neighborhood we have:

$$Ox \subseteq X \setminus W = \bigcup_t (X \setminus W) \cap X_t.$$

Hence $X \setminus W$ is open, and, by Theorem 2, the set $X \setminus W$ has weight $\leq \tau$.

Denote by B_0 a base of cardinality $\leq \tau$ of open sets in the subspace $X \setminus W$, and by B_x some base of the space X at the point x, of cardinality $\leq \tau$. Then

$$B = B_0 \cup \left(\bigcup_{x \in W} B_x \right)$$

is a base in X, and therefore we obtain that

$$\tau_1 \leq \operatorname{card} B \leq \max(\tau, \operatorname{card} W).$$

Thus W is a set of cardinality $\geq \tau_1$. Let Ox be any neighborhood of any point $x \in W$. Then Ox satisfies all the hypotheses of Theorem 3 relative to X, and, since we have proved that the cardinality of W is $> \tau$, applying the preceding arguments to the space Ox, we obtain that the cardinality of $Ox \cap W$ is also greater than τ. Thus W is τ_1-dense in itself. Theorem 3 is proved.

Lemma 2. *Let, in a space X of weight $\leq \tau$, there be given a set R' of subsets of the space X, containing some base of neighborhoods (open or not) for every point $x \in X$. Then there exists a set $R \subseteq R'$ of cardinality $\leq \tau$ having the same property.*

Proof. Let B be a base of cardinality $\leq \tau$ of open sets in the space X. For each pair $G_1, G_2 \in B$ we choose, if it exists, such a set $V_{G_1,G_2} \in R'$ that

$$G_1 \subseteq V_{G_1,G_2} \subseteq G_2.$$

The set of all selected V_{G_1,G_2} is the desired R. Clearly, the cardinality of the system R does not exceed τ.

We shall show that for every point $x \in X$ there is found in R a base of its neighborhoods. Indeed, let $x \in G_2 \in B$. Then there exists a neighborhood $V \in R'$ of the point x such that $V \subseteq G_2$, and there exists a set $G_1 \in B$ satisfying the condition $x \in G_1 \subseteq \overline{V}$. But since $G_1 \subseteq V \subseteq G_2$, there exists $V_{G_1,G_2} \in R$, $x \in G_1 \subseteq V_{G_1,G_2} \subseteq G_2$. Thus the set V_{G_1,G_2} is a neighborhood of the point x from R, lying in the arbitrarily chosen neighborhood of the point x

from B. It follows that the system R forms a base of the space X at the point x. Lemma 2 is proved.

Theorem 4. *Let a regular space X be the sum of a set, of cardinality $\leq \tau$, of its subspaces X_t, each of which has weight $\leq \tau$. Then, in order that the space*

X have weight $\leq \tau$, it is sufficient that, for all t and $x \in X_t$, neighborhoods Ox (open or not) of the point x in the subspace X_t such that the set $[Ox] \cap \operatorname{Fr} X_t$ has a base of neighborhoods of cardinality $\leq \tau$ in the subspace $[X \setminus X_t]$, form a base of neighborhoods of the point x in X_t.

Proof. Suppose the condition of the theorem is fulfilled. Denote by R'_t the set of all subsets $A \subseteq X_t$ for which $[A] \cap \operatorname{Fr} X_t$ has a base of neighborhoods of cardinality $\leq \tau$ in the subspace $[X \setminus X_t]$. Then, by Lemma 2, there exists a subset R_t of the set R'_t which has cardinality $\leq \tau$ and in which there is a base of neighborhoods for every point $x \in X_t$ of the space X_t. Then $R = \bigcup_t R_t$ is also a system of sets whose cardinality does not exceed τ. For an arbitrary $A \in R_t$, by the symbol $B_{A,t}$ we denote some base of neighborhoods of the set $[A] \cap \operatorname{Fr} X_t$ in the subspace $[X \setminus X_t]$, having cardinality $\leq \tau$. And let, finally, B be the set of all sets of the form $A \cup E$, where $A \in R_t$ and $E \in B_{A,t}$ for some t. The set B has cardinality $\leq \tau$.

We shall prove that B contains a base of neighborhoods of every point of the space X. First, if $Ox \in R_t$ is a neighborhood of the point $x \in X_t$ in the space X_t and $E \in B_{Ox,t}$, then $Ox \cup E$ is a neighborhood of the point x in X. If Ox is a neighborhood of the point x in X, then this is clear. In the contrary case $x \in [X \setminus X_t]$ and the set E is a neighborhood of the point x in $[X \setminus X_t]$. Then in the space X there exists a neighborhood U of the point x satisfying the conditions $U \cap X_t \subseteq Ox$ and $(U \cap [X \setminus X_t]) \subseteq E$. Hence $U \subseteq (Ox \cup E)$, and therefore $Ox \cup E$ is a neighborhood of the point x in the space X.

Let now U be an arbitrary neighborhood of the point $x \in X_t$ in the space X. There then exists $Ox \in R_t$ such that $x \in [Ox] \subseteq U$, and for some $E \in B_{Ox,t}$ we have $Ox \cup E \subseteq U$. Thus we have shown that B indeed contains a base of neighborhoods of every point $x \in X$. Hence it follows at once that the set B_0 of interiors of the sets from B is a base of open sets of the space X of cardinality $\leq \tau$. The theorem is proved.

Theorem 5. *Let a regular space X be the sum of a set, of cardinality $\leq \tau$, of its subspaces X_t, each of which has weight $\leq \tau$. Then, in order that the space X itself have weight $\leq \tau$, it is sufficient that, for all t and $x \in X_t$, there exist a neighborhood Ox of the point x in the space X such that the set $[Ox \setminus X_t]$ is bicompact.*

Proof. From the cited theorem of Arhangel' skiĭ it follows that in the space X bicompact sets have weight $\leq \tau$. If Ox is such a neighborhood of the point $x \in X_t$ in the space X that $[Ox \setminus X_t]$ is bicompact, then for all neighborhoods $O'x$ of the point x in the subspace X_t, whose closures are contained in Ox, there exists a base of neighborhoods of the set $[O'x] \cap \operatorname{Fr} X_t$ in the subspace $[X \setminus X_t]$ of cardinality $\leq \tau$. This follows from the fact that $[O'x] \cap \operatorname{Fr} X_t$ is a closed subset of the bicompact $[Ox \setminus X_t]$, whose weight is $\leq \tau$, and which is a neighborhood of the set $[O'x] \cap \operatorname{Fr} X_t$ in the subspace $[X \setminus X_t]$. Such neighborhoods $O'x$ of the point $x \in X_t$ obviously form a base of neighborhoods of the point x in X_t. Therefore, if for all t and $x \in X_t$ there exists a neighborhood Ox in X such that

$[Ox \setminus X_t]$ is a bicompact set, then, by Theorem 4, X has weight $\leq \tau$. Theorem 5 is proved.

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CITED LITERATURE

1. P. S. Aleksandrov, *Introduction to the general theory of sets and functions*, Moscow–Leningrad, 1948.

2. A. Arhangel' skiĭ, *DAN*, **126**, No. 2, 239 (1959).

Note: Figure translations are in progress. See original paper for figures.