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Abstract

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On a Stationary Solution for a System of Equations of Combustion Theory

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The one-dimensional process of combustion of a combustible gas mixture is described by the system (see (^{1}))

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(u)v, \\ \frac{\partial v}{\partial t} - \lambda \frac{\partial^2 v}{\partial x^2} &= -f(u)v. \end{aligned} \tag{1}$$

Here u is the temperature of the mixture; v is the concentration of the active substance; $f(u)v$ is the reaction rate; $f(u) = 0$ for $u < \alpha$; $f(u) > 0$ for $u > \alpha$; $\lambda = D\rho c/k$; D is the diffusion coefficient; ρ is the density of the substance; c is its heat capacity, k is the coefficient of thermal conductivity.

A solution of system (1) of the form

$$u = \tilde{u}(x + mt + C), \qquad v = \tilde{v}(x + mt + C), \tag{2}$$

where

$$m = \text{const} > 0, \qquad C = \text{const}, \qquad \tilde{u}(-\infty) = u_- < \alpha, \qquad \tilde{u}(+\infty) = u_+ > \alpha,$$

$$\tilde{v}(-\infty) = v_- > 0, \qquad \tilde{v}(+\infty) = 0,$$

is called stationary. For $\lambda = 1$, system (1) reduces to a single equation. The existence of a stationary solution for this equation was proved in (^{1}). There, the stationary solution for system (1) is also considered. In the present paper, the existence of the latter is proved for arbitrary $\lambda > 0$, and uniqueness for $0 < \lambda < 1$.

Let $\xi = x + mt$. Then the stationary solution will satisfy the system

$$m\frac{du}{d\xi} - \frac{d^2u}{d\xi^2} = f(u)v, \qquad m\frac{dv}{d\xi} - \lambda\frac{d^2v}{d\xi^2} = -f(u)v. \tag{1'}$$

This system has the first integral

$$m(u+v) - \frac{du}{d\xi} - \lambda\frac{dv}{d\xi} = C = \text{const}. \tag{3}$$

It is natural to assume that the stationary solution also satisfies the conditions

$$\frac{du}{d\xi}\bigg|_{\xi=\pm\infty} = \frac{dv}{d\xi}\bigg|_{\xi=\pm\infty} = 0. \tag{2'}$$

From (3), (2), and $(2')$ it follows that the numbers u_+, u_-, v_- must satisfy the additional restriction $u_+ = u_- + v_-$.

Thus, it is required to prove that, for some $m = m_0$, there exists a solution of system $(1')$ satisfying conditions (2) and $(2')$. Here it is assumed that $u_- < \alpha < u_+$, $v_- > 0$, $u_+ = u_- + v_-$. We replace the second equation of system $(1')$ by the integral (3), where $C = mu_+$ (by virtue of the boundary condi-

···wise). Put $du/d\xi = p$. We shall regard u as the independent variable. Then instead of system $(1')$ we obtain the system

$$\frac{dp}{du} = m - \frac{f(u)v}{p}, \qquad \frac{dv}{du} = \frac{m}{\lambda p}(u + v - u_+) - \frac{1}{\lambda}. \tag{4}$$

Instead of conditions (2) and $(2')$ we obtain the conditions

$$v\big|_{u=u_-} = v_-, \qquad v\big|_{u=u_+} = 0, \qquad p\big|_{u=u_-} = p\big|_{u=u_+} = 0. \tag{5}$$

It can be shown that through the singular point $u = u_+$, $v = 0$, $p = 0$ there pass three integral curves of system (4). Moreover, for two of them $p < 0$ for $u < u_+$ in a sufficiently small neighborhood of u_+, and for one $p > 0$ for the same u. Only the last of these curves is meaningful to consider. Let $p = p(u,m)$, $v = v(u,m)$ be its equations. From the second equation of system (4), where $p = p(u,m)$, we find

$$v(u,m) = u_+ - u + \left(\frac{1}{\lambda} - 1\right)\int_u^{u_+} \exp\left[-\frac{m}{\lambda}\int_u^{u_2}\frac{du_1}{p(u_1,m)}\right]du_2. \tag{6}$$

It follows from (6) that for $u < u_+$, in some neighborhood of u_+, the inequalities

$$u_+ - u < v < \frac{1}{\lambda}(u_+ - u) \qquad \text{for } \lambda < 1; \tag{7}$$

$$\frac{1}{\lambda}(u_+ - u) < v < u_+ - u \qquad \text{for } \lambda > 1. \tag{7'}$$

hold.

Let $p = p_1(u, m)$ be the separatrix of the saddle at the point $u = u_+$, $p = 0$ for the equation $dp/du = m - f(u)(u_+ - u)/p$, passing above the u-axis for $u < u_+$; let $p = p_2(u, m)$ be the separatrix of the same kind for the equation $dp/du = m - f(u)(u_+ - u)/\lambda p$. For $u < u_+$ the inequalities

$$p_1(u, m) < p(u, m) < p_2(u, m) \qquad \text{for } \lambda < 1; \tag{8}$$

$$p_2(u, m) < p(u, m) < p_1(u, m) \qquad \text{for } \lambda > 1. \tag{8'}$$

hold.

Indeed, we have, for example,

$$\frac{d\Delta p}{du} - \frac{f(u)(u_+ - u)}{pp_1}\Delta p = -\frac{f(u)}{p}\left[v - (u_+ - u)\right], \qquad \Delta p = p - p_1.$$

Solving this equation under the condition $\Delta p|_{u=u_+} = 0$ and using (7) and (7′), we obtain $p_1(u, m) < p(u, m)$ for $\lambda < 1$; $p(u, m) < p_1(u, m)$ for $\lambda > 1$. From inequalities (8) and (8′) it follows that the solution $p = p(u, m)$, $v = v(u, m)$ can be continued to the whole interval $u_- < u < u_+$. Direct calculation shows that the functions $p_1(u, m)$ and $p_2(u, m)$ are strictly positive for $m = 0$, $u_- \leq u < u_+$. As m increases, the quantities $p_1(u_-, m)$ and $p_2(u_-, m)$ decrease, and there exist such positive numbers m_1 and m_2(1) that $p_1(u_-, m_1) = 0$, $p_2(u_-, m_2) = 0$, $p_1(u, m_1) > 0$ for $u_- < u < u_+$; $p_2(u, m_2) > 0$ for $u_- < u < u_+$. Then, by virtue of (8) and (8′), there is an $m = m_0 > 0$ such that $p(u_-, m_0) = 0$, $p(u, m_0) > 0$ for $u_- < u < u_+$.

The equality $v(u_-, m_0) = v_-$ is then obtained automatically from (6). Thus, the curve $p = p(u, m_0)$, $v = v(u, m_0)$ will be the desired one. Solving the equation $du/d\xi = p(u, m_0)$, we find $u = \tilde{u}(\xi + C)$. $v = v(\tilde{u}(\xi + C), m_0) = \tilde{v}(\xi + C)$, where C is an arbitrary constant—the shift of the stationary wave along the ξ-axis.

We shall prove that for $\lambda < 1$ the number m_0 is determined uniquely, i.e., there is uniqueness of the stationary solution up to the shift C.

Suppose the contrary: let a solution of problem (4)–(5) exist for $m = m_1$ and for $m = m_2$, where $0 < m_1 < m_2$. Let p_1, v_1 be the solution corresponding to m_1; p_2, v_2 the solution corresponding to m_2.

Subtracting from the equality $\dfrac{dp_2}{du} = m_2 - \dfrac{f(u)v_2}{p_2}$ the equality $\dfrac{dp_1}{du} = m_1 - \dfrac{f(u)v_1}{p_1}$, and from the equality $\dfrac{dv_2}{du} = \dfrac{m_2}{\lambda p_2}(u + v_2 - u_+) - \dfrac{1}{\lambda}$ the equality $\dfrac{dv_1}{du} = \dfrac{m_1}{\lambda p_1}(u + v_1 - u_+) - \dfrac{1}{\lambda}$, we obtain

$$\frac{d\Delta p}{du} - \frac{f(u)v_1}{p_1 p_2}\Delta p = \Delta m - \frac{f(u)}{p_2}\Delta v; \tag{9}$$

$$\frac{d\Delta v}{du} - \frac{m_2}{\lambda p_2}\Delta v = \frac{u - u_+ + v_1}{\lambda p_1}\left(\Delta m - \frac{m_2}{p_2}\Delta p\right), \tag{10}$$

where

$$\Delta p = p_2 - p_1, \qquad \Delta v = v_2 - v_1, \qquad \Delta m = m_2 - m_1.$$

Considering equality (9) as a linear equation with respect to Δp, and equality (10) as a linear equation with respect to Δv, and taking into account that $\Delta p = \Delta v = 0$ at $u = u_+$, we obtain, for $u_- < u < u_+$,

$$\Delta p = -\int_u^{u_+}\left[\Delta m - \frac{f(w)}{p_2(w)}\Delta v\right]\exp\left[-\int_u^w \frac{f(z)v_1(z)}{p_1(z)p_2(z)}\,dz\right]dw; \tag{11}$$

$$\Delta v = -\int_u^{u_+}[w + v_1(w) - u_+]\,\frac{1}{\lambda p_1}\left(\Delta m - \frac{m_2 \Delta p}{p_2}\right)\exp\left[-\int_u^w \frac{m_2\,dz}{\lambda p_2(z)}\right]dw. \tag{12}$$

Investigating the singular point $u = u_+$, $v = 0$, $p = 0$ of system (4), one can show that, as $u \to u_+ - 0$,

$$p(u, m) = k(u_+ - u) + o(u_+ - u), \qquad \text{where } k = \sqrt{\frac{m^2}{4\lambda^2} + \frac{f(u_+)}{\lambda}} - \frac{m}{2\lambda},$$

$$v(u, m) = l(u_+ - u) + o(u_+ - u), \qquad \text{where } l = \frac{m + k}{m + \lambda k}.$$

Hence, from the inequality $dk/dm < 0$ and the inequality $dl/dm < 0$, valid for $\lambda < 1$, it follows that

$$\Delta p < 0, \qquad \Delta v < 0 \quad \text{for} \quad u_+ - \delta < u < u_+, \tag{13}$$

where $\delta > 0$ is sufficiently small. In view of (6), $u + v_1(u) - u_+ > 0$ for $\lambda < 1$, $u_- < u < u_+$.

Hence, from equalities (11) and (12) it follows that inequalities (13) can be continued to the whole half-segment $u_- \leq u < u_+$. But this contradicts the assumption that, at $u = u_-$, the equalities $p_1 = p_2 = 0$, $v_1 = v_2 = v_-$ hold.

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CITED LITERATURE:

1. Ya. B. Zel' dovich, ZhFKh, **22**, No. 1, 27 (1948).

Note: Figure translations are in progress. See original paper for figures.