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Abstract

Full Text

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THE HYPOTHESIS OF THE TRAJECTORY OF A QUANTUM PARTICLE

(Presented by Academician M. A. Leontovich, 22 VI 1963)

As is well known, the concept of a continuous trajectory and the simultaneous existence of coordinate and momentum contradict the postulates of quantum mechanics. Here one has in mind a classical trajectory $x(t)$, which has at least two derivatives: $\dot{x}(t) = p(t)$—the velocity or momentum, if the particle mass is unity, and $\ddot{x}(t) = \dot{p}(t) = w(t)$—the acceleration.

Let us consider the hypothesis of a quantum particle with a continuous, but nonclassical, trajectory $x(t)$, having a first derivative $\dot{x}(t) = p(t)$, but not having a second derivative. Such a trajectory may possess a remarkable property: $p(t)$ may be discontinuous everywhere. The purpose of the present article is to indicate the mathematical conditions that such a nonclassical trajectory must satisfy, if it exists.

In quantum mechanics the state of motion of a quantum particle is described by the ψ-function. In the "coordinate" representation the arguments of the ψ-function are x and t; in the momentum representation, p, t. The ψ-function satisfies the Schrödinger equation

$$i\hbar\partial\psi/\partial t = H\psi. \tag{1}$$

The "coordinate-momentum," or "phase," representation has also long been known; in it, according to Wigner, the state of motion is described by a function $f(t, x, p)$, related to $\psi(t, x)$ by

$$f(t, x, p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi^+\left(t, x - \frac{\hbar\xi}{2}\right) e^{-ip\xi}\psi\left(t, x + \frac{\hbar\xi}{2}\right) d\xi. \tag{2}$$

$f(t, x, p)$ may be regarded as an integral of motion for the quantum particle; for the latter, differentiating f with respect to time and using (1), we obtain an equation of the form

$$A_k[f(t, x, p)] = \left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial x} + \int_{-\infty}^{+\infty} R(p - p', x)\, dp'\right) f(t, x, p) = 0, \tag{3}$$

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where the kernel of the integral operator R is an antisymmetric function of $\zeta = p - p'$:

$$R(\zeta, x) = -\frac{i}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\xi\zeta}\, d\xi \left[u\left(x + \frac{\hbar\xi}{2}\right) - u\left(x - \frac{\hbar\xi}{2}\right)\right], \qquad (4)$$

$\hbar$ is Planck' s constant, and $u(x)$ is the potential energy of the field.

As $\hbar \to 0$, the integral operator in A_k, in accordance with the correspondence principle, becomes in the limit

$$\lim \int_{-\infty}^{+\infty} R(p - p', x)\, dp'\, f(t, x, p') = -\frac{\partial u(x)}{\partial x} \frac{\partial f(t, x, p)}{\partial p}$$

and, consequently, the classical operator A, as it should be,

$$Af = 0; \qquad A = \lim_{\hbar \to 0} A_k = \partial/\partial t + p\, \partial/\partial x + w\, \partial/\partial p; \qquad w = -\partial u(x)/\partial x. \quad (5)$$

Equations (2), (3), (4) are equivalent to the Schrödinger equation (1) in the sense that the former can be obtained from the latter, and conversely, from (2), (3), (4) one can obtain the Schrödinger equation for the ψ-function.

Let $f(t, x, p)$ be an integral of motion, by definition satisfying the condition of invariance with respect to an infinitesimal transformation. Suppose that there exists a nonclassical, i.e., not possessing the first two derivatives, function $x(t)$, compatible with (3). Assuming

the existence of the latter, let us find the conditions which such a trajectory must satisfy.

In the quantum-mechanical operator A_k, just as in the classical A (5), there are identical terms containing derivatives with respect to t and x, and only the term with the integral operator with respect to p is replaced by a term containing a derivative with respect to p. Therefore one should think that the nonclassical trajectory $x(t)$ is continuous and has the first derivative $\dot{x}(t) = p(t)$. However, the quantum trajectory, if it exists, has no second derivative with respect to x, i.e., no first derivative of $p(t)$, or acceleration.

Such a conception does not contradict our “visual”idea of the motion of a physical particle along a continuous trajectory, since a discontinuity of $x(t)$ would mean an instantaneous displacement of the particle by a finite distance. Conversely, instantaneous increments of the particle' s momentum do not contradict the “visual” idea of the quantum nature of the transfer of momentum–energy in finite portions.

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Assuming $x(t)$ to be a continuous function of t, having the first derivative $\dot{x}(t) = p(t)$, and assuming that $p(t)$ has no derivative, for arbitrarily small τ we expand $f(t{+}\tau, x{+}\Delta x, p{+}\Delta p)$ in a series only in τ and Δx, and, discarding infinitesimals of higher orders, write

$$f(t,x,p) = f(t+\tau, x+\Delta x, p+\Delta p) = \\ = \tau\,\partial f/\partial t + \Delta x\,\partial f/\partial x + f(t,x,p+\Delta p) + \text{ i.s.h.o.} \tag{6}$$

Since $p(t)$ has no derivative, the operation of dividing the last equation by τ and passing to the limit as $\tau \to 0$, which leads to (5), loses its meaning.

Let us generalize the operation of passage to the limit as follows: integrate (6) over the intervals $(0, +s)$ and $(0, -s)$ and subtract the second result from the first (s arbitrarily small):

$$\int_0^s f(t,x,p)\,d\tau - \int_{-s}^0 f(t,x,p)\,d\tau =$$

$$= \int_0^s \tau\,\frac{\partial f}{\partial t}\,d\tau - \int_{-s}^0 \tau\,\frac{\partial f}{\partial t}\,d\tau + \int_0^s \Delta x(\tau)\,\frac{\partial f}{\partial x}\,d\tau - \int_{-s}^0 \Delta x(\tau)\,\frac{\partial f}{\partial x}\,d\tau +$$

$$+ \int_0^s f(t,x,p+\Delta p)\,d\tau - \int_{-s}^0 f(t,x,p+\Delta p)\,d\tau + \text{ i.s.h.o.}$$

Taking outside the integral sign the integrand expressions that do not depend on τ, carrying out the evident calculations, and dividing by s^2, we obtain, for $\Delta x(\tau) = \dot{x}(t)\tau + \text{ i.s.h.o.} = \tau p(t) + \text{ i.s.h.o.}$,

$$0 = \frac{\partial f}{\partial t} + p\,\frac{\partial f}{\partial x} + \frac{1}{s^2}\int_0^s [\, f(p+\Delta p(\tau)) - f(p+\Delta p(-\tau))\,]\,d\tau + \text{ i.s.h.o.}$$

We now introduce notation for the linear operator L_s and, with its help, write the invariance condition for $f(t,x,p)$ more briefly:

$$L_s f(t,x,p) = \frac{1}{s^2}\int_0^s [\, f(t,x,p+\Delta p(\tau)) - f(t,x,p+\Delta p(-\tau))\,]\,d\tau,$$

$$A_s f = \partial f/\partial t + p\,\partial f/\partial x + L_s f. \tag{7}$$

For the case when the derivative of $p(t)$ exists in an arbitrarily small interval $(0, s)$, $\Delta p(\tau) = \tau\dot{p}(t) + \text{ i.s.h.o.}$. Therefore, in the limit as $s \to 0$, $L_s f \to \dot{p}\,\partial f/\partial p$

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and, consequently, $\lim_{s\to 0} A_s = A$, i.e., in accordance with the correspondence principle, for a classical trajectory we also obtain the classical operator (5).

Suppose that for a nonclassical trajectory $x(t)$, i.e., when the derivative of $p(t)$ does not exist, nevertheless there exists the limit of the operator L_s as $s \to 0$, i.e., suppose that there exists

$$Lf = \lim_{s\to 0} L_s f = \lim_{s\to 0} \frac{1}{s^2} \int_0^s [f(t,x,p+\Delta p(\tau)) - f(t,x,p+\Delta p(-\tau))]\, d\tau. \tag{8}$$

In this hypothetical case, (8), in the limit, will give equation (7) the form

$$\partial f/\partial t + p\,\partial f/\partial x + Lf = 0. \tag{9}$$

If the operator Lf (8) can take the form of the integral operator

$$\int_{-\infty}^{+\infty} R(p-p',x)\, dp'$$

in the quantum-mechanical operator (3), (4), then the proposed generalization of the limiting operation will lead us to the goal; that is, the condition for the existence of a nonclassical quantum trajectory will reduce to the condition for the existence of such a twice nondifferentiable function $x(t)$, for which at any instant of time t there exists an operator equal to the operator (3), (4).

What is the form of the operator L for a nonclassical trajectory, if it exists?

Let $f(p)$ be some function of p, and let, in turn, $p(t)$ be a function of the argument t, and let $\Delta(t,\tau) = p(t+\tau) - p(t)$ be the increment of the function p at the point t when the argument is increased by τ. If $f(p)$ has all derivatives, then, according to (8),

$$L_s f = \frac{1}{s^2} \int_0^s [f(p+\Delta p(\tau)) - f(p+\Delta p(-\tau))]\, d\tau =$$

$$= \sum_1^{\infty} \frac{1}{n!} \frac{\partial f^n}{\partial p^n} \frac{1}{s^2} \int_0^s [\Delta^n(t,\tau) - \Delta^n(t,-\tau)]\, d\tau. \tag{10}$$

Suppose that there exists a countable set of limits

$$L^{(n)} p = \lim_{s\to 0} \frac{1}{s^2} \int_0^s [\Delta^n(t,\tau) - \Delta^n(t,-\tau)]\, d\tau, \qquad n = 1,2,3,\dots. \tag{11}$$

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Assuming, moreover, that the series (10) converges and that $L^{(n)}p$ exists for any value of t, we find that

$$Lf = \sum_1^\infty \frac{1}{n!}\frac{\partial f^n}{\partial p^n} L^{(n)}p. \tag{12}$$

Let $p(t)$ have no derivative; we shall suppose that there exists a class of functions, which we shall call functions having a fluke, if for the function $p(t)$ at each t there exists the countable set of limits (11).

Let us now return to the integral operator in p with kernel (4) in the quantum-mechanical operator (3) A_k; expanding this integral operator of the function f in a series, we obtain

$$\int_{-\infty}^{+\infty} R(p-p',x)\,dp'\,f(t,x,p') = \int_{-\infty}^{+\infty} R(\xi,x)\,d\xi\,f(t,x,p+\xi) =$$

$$= \int_{-\infty}^{+\infty} R(\xi,x)\,d\xi \sum_0^\infty \frac{1}{n!}\frac{\partial f^n}{\partial p^n}\xi^n = \sum_1^\infty \frac{\overline{\xi^n}(x)}{n!}\frac{\partial f^n}{\partial p^n}, \qquad \overline{\xi^n}(x) = \int_{-\infty}^{+\infty} \xi^n R(\xi,x)\,d\xi. \tag{13}$$

We note that, by virtue of the antisymmetry $R(\xi,x) = -R(-\xi,x)$, only the odd terms of the series are different from zero.

Comparing (13) with (12), it is not difficult to notice that the operator Lf coincides with the integral term of the quantum-mechanical operator A_k, if for every n

$$L^{(n)}p = \overline{\xi^n}(x) = \left(\frac{(-i)^{n+1}\hbar^{n-1}}{2^n}\right)[1-(-1)^n]\,(\partial^n u(x)/\partial x^n). \tag{14}$$

One can avoid expanding the function f in a series if one uses the following formula, which follows from the properties of the Fourier integral: if f is a function having a finite number of maxima and minima in any finite interval and is absolutely integrable on the interval $(-\infty,\infty)$, then

$$f(p+\Delta p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dp'\,f(p')\int_{-\infty}^{+\infty} e^{i\xi(p-p'+\Delta(\tau))}\,d\xi. \tag{15}$$

Even if $\Delta(t,\tau)$ is discontinuous everywhere on the interval $(0,s)$, then, integrating in the Lebesgue sense, we write, changing the order of integration:

$$\int_0^s f(p+\Delta p(t,\tau))\,d\tau = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} dp'\,f(p')\,d\xi\,e^{i\xi(p-p')}\int_0^s e^{i\xi\Delta(t,\tau)}\,d\tau. \tag{16}$$

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Using the latter, Lf can be represented in the following form (making use of (8)):

$$Lf = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} dp'\, f(t,x,p')\, e^{i\xi(p-p')}\, i\xi\, d\xi\, Lp, \tag{17}$$

where

$$Lp = \lim_{s\to 0} \frac{1}{i\xi s^2} \int_0^s \left[e^{i\xi\Delta p(t,\tau)} - e^{i\xi\Delta p(t,-\tau)}\right] d\tau.$$

Comparing the last expressions with the integral operator in the quantum-mechanical equation (3), (4), we obtain the equation for the locus of $p(t)$

$$Lp = -(1/\hbar\xi)\,[u(x+\hbar\xi/2) - u(x-\hbar\xi/2)]. \tag{18}$$

Now we can answer the question we have posed. A quantum-mechanical operator of the form (3), (4) is compatible with the interpretation of a nonclassical trajectory $x(t)$, nowhere possessing a second derivative, if there exists such a function $x(t)$ that satisfies the trajectory equation

$$L\dot{x}(t) = -(1/\hbar\xi)\,[u(x+\hbar\xi/2) - u(x-\hbar\xi/2)].$$

More precisely: in order for a quantum particle to have a trajectory $x(t)$, it is necessary:

1) For any t there must exist the locus of $\dot{x}(t)$, i.e. there must exist a function $Lp = \varphi(t,\xi)$. In the trivial case, when $x(t)$ has a second derivative, $L\dot{x}(t) = \ddot{x}(t)$, i.e. $\varphi(t,\xi)$ is a function of the single argument t only.
2) The function $\varphi(t,\xi) = Lp$, which may be called the quantum acceleration, by analogy with classical mechanics, must be connected with the expression for the potential energy $u(x)$ by relation (18).

In other words, Newton' s equation for the trajectory

$$\ddot{x}(t) = -\partial u(x)/\partial x \tag{19}$$

is replaced by the equation

$$L\dot{x}(t) = -(1/\hbar\xi)\,[u(x+\hbar\xi/2) - u(x-\hbar\xi/2)]. \tag{20}$$

If $x(t)$ has a second derivative, then the last equation passes into the Newtonian one at $\hbar = 0$.

If there exists a function $x(t)$ that has no second derivative and satisfies the last equation, then the quantum particle has a trajectory, and conversely.

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Note: Figure translations are in progress. See original paper for figures.

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