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Abstract

Full Text

MATHEMATICS

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CONFORMAL MAPPINGS OF A MULTI-CONNECTED DOMAIN ONTO MULTI-SHEETED CANONICAL SURFACES

(Presented by Academician V. I. Smirnov, 9 I 1963)

§ 1. Let G be a bounded m -connected domain of the z -plane with boundary C , consisting of simple closed analytic curves C_1, \dots, C_m ; ξ_1, \dots, ξ_s , $s \geq 1$, be arbitrary mutually distinct points of the domain G ;

$$\alpha_{0,j}, \alpha_{1,j}, \dots, \alpha_{p_j,j}, \quad j = 1, \dots, s,$$

be arbitrary coefficients with $\sum_{j=0}^s \alpha_{0,j} = 0$, but not all equal to zero (an admissible system of coefficients);

$$S(z; \xi, \alpha) = \sum_{j=1}^s \left[\sum_{k=1}^{p_j} \frac{\alpha_{k,j}}{(z - \xi_j)^k} + \alpha_{0,j} \log(z - \xi_j) \right].$$

Theorem 1. For any prescribed angle θ , $-\pi/2 < \theta \leq \pi/2$, and any prescribed function $S(z; \xi, \alpha)$, there exists a function, unique up to an additive constant,

$$\Phi_\theta(z) = S(z; \xi, \alpha) + F_\theta(z),$$

with $F_\theta(z)$ regular* in the closed domain G , which maps each boundary component of the domain G onto segments of certain straight lines making angle θ with the real axis.

Corollary. Let ξ_1, \dots, ξ_s , $s \geq 2$, be arbitrary prescribed mutually distinct points of the domain G ; let m_1, \dots, m_s be arbitrary prescribed positive integers such that

$$\sum_{j=1}^k m_j = \sum_{j'=k+1}^s m_{j'};$$

let θ be any prescribed angle, $-\pi/2 < \theta \leq \pi/2$. Then there exists a function $\Psi_\theta(z)$, regular in the domain G except for poles at the points $z = \xi_{j'}$, respectively

of orders $m_{j'}$, $j' = k + 1, \dots, s$, with an expansion in a neighborhood of $z = \xi_s$ of the form

$$\Psi_\theta(z) = \frac{1}{(z - \xi_s)^{m_s}} + \dots,$$

having in this domain zeros only at the points ξ_j , respectively of multiplicities m_j , $j = 1, \dots, k$, and mapping each boundary component of it onto an arc of a certain logarithmic spiral making angle θ with the rays issuing from the origin; such a function is unique.

We extend the notion of the exterior area of a meromorphic function ⁽¹⁾ to functions of the form $f(z) = S(z; \xi, \alpha) + g(z)$, where the function $g(z)$ is regular in the domain G . Namely, by the exterior area $\bar{A}(f)$ of the function $f(z)$ in the domain G we shall mean the limit (finite or infinite)

$$\bar{A}(f) = - \lim_{\nu \rightarrow \infty} \frac{1}{2i} \int_{C^{(\nu)}} f'(z) \overline{f(z)} dz,$$

where $C^{(\nu)}$ are the boundaries of domains $G^{(\nu)}$ which, as $\nu \rightarrow \infty$, approximate the domain G from within.

We introduce for consideration the functions

$$P(z) = \frac{1}{2} [\Phi_{\pi/2}(z) - \Phi_0(z)], \quad Q(z) = \frac{1}{2} [\Phi_{\pi/2}(z) + \Phi_0(z)],$$

corresponding to the prescribed function $S(z; \xi, \alpha)$.

* By regular and meromorphic functions we mean single-valued regular and meromorphic functions.

It is easy to see that $A(P) = \bar{A}(Q)$, where $A(P)$ is the area of the image of the domain G under the mapping $w = P(z)$.

Theorem 2. *In the class of all functions of the form $f(z) = S(z; \zeta, a) + g(z)$, where $S(z; \zeta, a)$ is a fixed function and $g(z)$ is an arbitrary function regular in the domain G , the largest outer area is attained only by the function $Q(z)$.*

We shall regard the points ζ_1, \dots, ζ_s as fixed and consider all possible admissible systems of coefficients $\alpha_{0,j}, \alpha_{1,j}, \dots, \alpha_{p_j,j}$, $j = 1, \dots, s$. To each such system of coefficients there corresponds a function $P(z)$, which we now normalize by the condition $P(\zeta_s) = 0$ and denote by $P(z; \zeta_s)$.

Lemma. *For any prescribed system of constants $\beta_j^{(0)}, \beta_j^{(1)}, \dots, \beta_j^{(p_j)}$, $j = 1, \dots, s$, $\beta_s^{(0)} = 0$, not all zero simultaneously, there exists a unique system of coefficients*

$\alpha_{0,j}, \alpha_{1,j}, \dots, \alpha_{p_j,j}$, $j = 1, \dots, s$, for which the corresponding function $P(z; \zeta_s)$ satisfies the conditions

$$P^{(k)}(\zeta_j; \zeta_s) = \beta_j^{(k)}, \quad k = 0, 1, \dots, p_j, \quad j = 1, \dots, s.$$

We denote the function $P(z; \zeta_s)$ satisfying the conditions of the lemma by $P_*(z; \zeta_s)$.

Theorem 3. *Let any mutually distinct points ζ_1, \dots, ζ_s of the domain G and any constants $\beta_j^{(0)}, \beta_j^{(1)}, \dots, \beta_j^{(p_j)}$, $j = 1, \dots, s$, $\beta_s^{(0)} = 0$, not all zero, be given. In the class of all functions $g(z)$, regular in the domain G and satisfying the conditions $g^{(k)}(\zeta_j) = \beta_j^{(k)}$, $k = 0, 1, \dots, p_j$, $j = 1, \dots, s$, the least value of the area $A(g)$ is attained only by the function $P_*(z; \zeta_s)$.*

It is clear that this theorem also immediately yields the solution of the problem on the minimum area in the class of all functions regular in a given finitely connected domain and having, at prescribed points of it, arbitrary prescribed initial segments of their Taylor expansions. This problem was previously solved for a multiply connected domain in the class of regular functions with fixed values at two prescribed points of the domain ⁽²⁾, and in the class of regular functions with fixed values of several first derivatives at a prescribed point of the domain ⁽¹⁾.

§ 2. Denote by $\mathfrak{L}_{\zeta,a}(G)$ the class of all functions having the form $f(z) = S(z; \zeta, a) + g(z)$, with fixed function $S(z; \zeta, a)$, arbitrary function $g(z)$ regular in the domain G , and satisfying the condition $\bar{A}(f) \geq 0$. Examples of functions of this class are the functions $\Phi_\theta(z)$ and $Q(z)$. The class $\mathfrak{L}_{\zeta,a}(G)$ turns out to be convex. Let some class $\bar{\mathfrak{L}}_{\zeta,a}(G)$ be considered. Then, for any function of the form

$$f(z) = \sum_{j=1}^s \left[\sum_{k=1}^{p_j} \frac{\gamma_{k,j}}{(z - \zeta_j)^k} + \gamma_{0,j} \log(z - \zeta_j) \right] + F(z),$$

where $\{\gamma\}$ are arbitrary coefficients and $F(z)$ is an arbitrary function regular in the domain G , the functional

$$I(f) = \sum_{j=1}^s \left[\alpha_{0,j} F(\zeta_j) - \sum_{k=1}^{p_j} \frac{\alpha_{k,j}}{(k-1)!} F^{(k)}(\zeta_j) \right]$$

is defined.

In particular, $A(P) = \pi I(P)$.

Let $v_\nu(z)$, $\nu = 1, 2, \dots$, be an orthonormal system of functions of the class $\mathcal{H}(G)$ (see ⁽¹⁾), complete in this class. Setting $(f', v_\nu) = -\frac{1}{2i} \int_C f' \bar{v}_\nu d\bar{z}$, we have the theorem:

Theorem 4. *If the function $f(z) \in \mathfrak{L}_{\zeta,a}(G)$ and its regular part is regular in the closed domain G , then we have the sharp estimate*

$$\sum_{\nu=4}^{\infty} |(f', v_{\nu})|^2 \leq \pi I(P).$$

Theorem 5. For any fixed θ , $-\pi/2 < \theta \leq \pi/2$, in the class $\overline{\mathfrak{L}}_{\zeta, \alpha}(G)$ the extremal problem

$$\operatorname{Re}\{e^{-2i\theta} I(f)\} = \min$$

is solved only by the function $\Phi_{\theta}(z)$.

Theorem 6. If the function $f(z)$ ranges over the class $\overline{\mathfrak{L}}_{\zeta, \alpha}(G)$, then the range of values of the functional $I(f)$ is the disk

$$|w - I(Q)| \leq I(P),$$

and to each point on the boundary of this disk there corresponds only the function $\Phi_{\theta}(z)$ with a suitably determined θ .

Theorem 7. In the subclass of all functions $f(z)$ from $\overline{\mathfrak{L}}_{\zeta, \alpha}(G)$ with any fixed value of $\operatorname{Re}\{I(f)\}$, the greatest exterior area in the domain G is attained only by the function $Q(z) + \lambda P(z)$ of this subclass, with a unique λ from the segment $[-1, 1]$.

§ 3. The results of the preceding paragraphs can be applied to the study of functions meromorphic and p -valent on the average in the domain G . Denote by $\overline{\Sigma}_p(\zeta_1, \dots, \zeta_k; \zeta)$ the class of all functions $f(z)$, regular in the domain G except for a p -fold pole at its point ζ , with expansion in its neighborhood of the form

$$f(z) = \frac{1}{(z - \zeta)^p} + \dots,$$

having zeros at the points of the domain ζ_j , $j = 1, \dots, k$, respectively of multiplicities m_j , with

$$\sum_{j=1}^k m_j = p,$$

and p -valent on the average in the domain G in the following sense: for any $r > 0$, the area of that part of the image of the domain G under the mapping $w = f(z)$ which lies over the disk $|w| < r$ does not exceed $p\pi r^2$. By $\overline{\Sigma}_p^*(\zeta_1, \dots, \zeta_k; \zeta)$ denote the subclass of those functions from $\overline{\Sigma}_p(\zeta_1, \dots, \zeta_k; \zeta)$ for which $\arg f(z)$ has single-valued branches in neighborhoods, lying in the domain G , of its boundary components.

Theorem 8. If the function $f(z)$ ranges over the class $\overline{\Sigma}_p^*(\zeta_1, \dots, \zeta_k; \zeta)$, then the range of values of the functional

$$\sum_{j=1}^k m_j \log \frac{f^{(m_j)}(\zeta_j)}{q^{(m_j)}(\zeta_j)}$$

is the disk

$$|w| \leq \sum_{j=1}^k m_j P(\zeta_j; \zeta),$$

and to each point on the boundary of this disk there corresponds only the function $\Psi_\theta(z)$ with suitably determined θ . Here $\Psi_\theta(z)$ is the derived function from Theorem 1, determined by $s = k + 1$ points ζ_j , $\zeta_s = \zeta$, and coefficients m_j , $m_s = -p$, $j = 1, \dots, k$; $P(z; \zeta)$ corresponds to the function

$$S(z; \zeta, a) = \sum_{j=1}^s a_{0,j} \log(z - \zeta_j)$$

with $a_{0,j} = m_j$, $j = 1, \dots, k$, $a_{0,s} = -p$; $q(z)$ is that one of the single-valued branches in G of

$$\sqrt{\Psi_{\pi/2}(z) \Psi_0(z)}$$

for which, in a neighborhood of $z = \zeta$, we have

$$q(z) = \frac{1}{(z - \zeta)^p} + \dots;$$

at the points ζ_j one takes the values of that branch of

$$\log \frac{f(z)}{q(z)}$$

which tends to zero as $z \rightarrow \zeta$.

In the case when the domain G is the unit disk, all the results obtained become completely explicit. In particular, we note the theorem:

Theorem 9. In the class of all functions

$$f(z) = z^p \left(1 + \frac{a_1}{z} + \dots \right),$$

p -valent and regular in the domain $|z| > 1$, except for the pole at $z = \infty$,

having at the fixed point $z_0 \neq \infty$ of this domain an expansion of the form

$$f(z) = f(z_0) + \frac{f^{(p)}(z_0)}{p!} (z - z_0)^p + \dots, \text{ the range of values of the functional } \log \frac{f^{(p)}(z_0)}{p!} \text{ is the circle } |w| \leq -p \log \left(1 - \frac{1}{|z_0|^2} \right).$$

In the case $p = 1$ this theorem turns out to be true for any fixed point z_0 of the domain $|z| > 1$, and we obtain a result known for single-valued functions ⁽³⁾.

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Note: Figure translations are in progress. See original paper for figures.

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