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Abstract

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ON THE MINIMAL RADICAL CLASS OVER THE CLASS OF ABELIAN GROUPS

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§ 1. The concept of a radical class of groups was introduced in the work (^{1}). Let the class of groups M be closed with respect to homomorphic images. The minimal radical class $R_0(M)$ generated by the class M can be defined as follows (^{3}). We shall call the groups in M groups of the 1st stage. If, for all ordinal numbers α smaller than a given β, groups of the α-th stage have already been defined, then a group G will be a group of the β-th stage over M if it has an ascending invariant series all of whose factors are groups of some α-th stages, $\alpha < \beta$. The radical class $R_0(M)$ consists of those and only those groups that have some stage over M. If M is the class of abelian groups, then the groups of the 2nd stage are the RI^*-groups (^{4}). In the work (^{1}) A. G. Kurosh posed the question of the radicality of the class of RI^*-groups. On the other hand, in (^{2}) it was shown that in this case an $R_0(M)$-group is precisely a subsolvable group of R. Baer (^{5}). Consequently, the question indicated above is the question from R. Baer' s work (^{5}) on the coincidence of the classes of subsolvable groups and RI^*-groups.

The purpose of the present work is to construct an example that gives a negative answer to this question. In constructing the example we rely on the example of I. D. Ado (^{6}) and on the related results of D. McLain (^{7}).

§ 2. **Example.** Let $P = \{\alpha, \beta, ...\}$ be the set of all rational numbers, and let K be an arbitrary field. Let E be the vector space over the field K with basis $e_{\alpha,\beta}$ for all $\alpha < \beta$, $\alpha, \beta \in P$. We introduce multiplication on E:

$$e_{\alpha,\beta} e_{\gamma,\delta} = \begin{cases} e_{\alpha,\delta}, & \text{if } \beta = \gamma, \\ 0 & \text{in all other cases.} \end{cases} \tag{1}$$

Then E will be an associative algebra. Adjoin externally to E the unit 1 of the field K and consider the set G of all elements of the form $1 + u$, $u \in E$, i.e. elements of the form

$$g = 1 + \sum_{i=1}^{n} a_{\alpha_i,\beta_i} e_{\alpha_i,\beta_i}, \qquad \alpha_i < \beta_i, \quad \alpha_i, \beta_i \in P, \quad a_{\alpha_i,\beta_i} \in K.$$

Lemma 1 (see (^{7})). *Every element $g \in G$ is a product of a finite number of elements of the form $1 + ae_{\alpha,\beta}$, where $\alpha < \beta$, $\alpha, \beta \in P$, $a \in K$.*

It is hence easy to see that G will be a group under multiplication, since multiplication is associative in view of the associativity of multiplication on E, and for every element $g \in G$ there exists an inverse element in view of Lemma 1 and in view of the fact that

$$(1 + ae_{\alpha,\beta})^{-1} = 1 - ae_{\alpha,\beta}, \qquad \alpha < \beta, \quad \alpha, \beta \in P, \quad a \in K.$$

This group was first considered by I. D. Ado in the work (^{6}).

Lemma 2 (see (^{7})). *If N is a nonidentity normal divisor of the group G, then for some α, β from P, $\alpha < \beta$, for all $a \in K$, N contains the elements $1 + ae_{\alpha,\beta}$.*

Lemma 3. For any rational number ξ, the subgroup N_ξ, generated by all generators of the form $1 + ae_{\alpha,\beta}$, where $a \in K$, $\alpha < \xi < \beta$, is an abelian normal divisor of the group G.

Indeed, if $1 + ae_{\alpha_1,\beta_1}$, $1 + be_{\alpha_2,\beta_2} \in N_\xi$, then $\alpha_1 < \xi < \beta_1$, $\alpha_2 < \xi < \beta_2$. Therefore, in view of condition (1), we obtain

$$[1 + ae_{\alpha_1,\beta_1},\, 1 + be_{\alpha_2,\beta_2}] = 1.$$

Consequently, N_ξ is abelian. On the other hand, if $1 + ae_{\alpha,\beta}$, $1 + be_{\gamma,\lambda}$ are two arbitrary generators of the group G, then

$$[1 + ae_{\alpha,\beta},\, 1 + be_{\gamma,\lambda}] = \begin{cases} 1 + abe_{\alpha,\lambda}, & \text{if } \beta = \gamma, \\ 1 - abe_{\gamma,\beta}, & \text{if } \alpha = \lambda, \\ 1, & \text{if } \beta \neq \gamma, \ \alpha \neq \lambda. \end{cases}$$

Therefore, if $1 + ae_{\alpha,\beta}$ is an arbitrary generator of the group N_ξ, and $1 + be_{\gamma,\lambda}$ is an arbitrary generator of the group G, then in all three indicated cases one has

$$[1 + ae_{\alpha,\beta},\, 1 + be_{\gamma,\lambda}] \in N_\xi.$$

Consequently, N_ξ is invariant in the group G. Lemma 3 is proved.

Since for an arbitrary generating element $1 + ae_{\alpha,\beta}$ of the group G there is a rational number ξ such that $\alpha < \xi < \beta$, it follows from Lemma 3 that $1 + ae_{\alpha,\beta}$ belongs to the abelian invariant subgroup N_ξ of the group G. Hence it follows that G is generated by its abelian invariant subgroups and, consequently, G is an RI^*-group.

Consider a certain subgroup of the group of all automorphisms of the group G. For an arbitrary rational number μ, consider the automorphism ψ_μ of the group G, defined as follows:

$$(1 + ae_{\alpha,\beta})\psi_\mu = 1 + ae_{\alpha+\mu,\beta+\mu}$$

for every generator $1 + ae_{\alpha,\beta}$ of the group G.

For any positive rational number δ, consider the automorphism φ_δ of the group G, defined as follows:

$$(1 + ae_{\alpha,\beta})\varphi_\delta = 1 + ae_{\alpha\delta,\beta\delta}.$$

Let Γ be the subgroup of the group of all automorphisms of the group G, generated by all ψ_μ and φ_δ for arbitrary rational μ and arbitrary rational $\delta > 0$. In the group Γ, the set of all ψ_μ forms a subgroup Ψ, isomorphic to the additive group of rational numbers, and the set of all φ_δ forms a subgroup Φ, isomorphic to the multiplicative group of positive rational numbers. Let us note that the intersection of the subgroups Ψ and Φ consists only of the identity automorphism $\psi_0 = \varphi_1$.

Between the elements of the subgroups Ψ and Φ there is the relation

$$\varphi_\delta^{-1}\psi_\mu\varphi_\delta = \psi_{\delta\mu}.$$

It follows that Ψ is invariant in the group Γ, and the group Γ is a split extension of the group Ψ by means of the group Φ. Consequently, Γ is a soluble group.

Denote by H the split extension of the RI^*-group G by means of its soluble group of automorphisms Γ. In the group H there therefore exist elements v_μ, u_δ $(\delta > 0)$, inducing respectively in G the automorphisms ψ_μ, φ_δ; these elements generate in H a subgroup isomorphic to Γ.

Lemma 4. The group G contains no nonunit proper subgroups invariant in the whole group H.

First we show that for two arbitrary generators $1 + ae_{\alpha,\beta}$ and $1 + ae_{\gamma,\lambda}$, $\alpha < \beta$, $\gamma < \lambda$, $a \in K$, of the group G, there exists an element of the group H whose transformation of the element $1+ae_{\alpha,\beta}$ gives $1+ae_{\gamma,\lambda}$. Indeed, let $1+ae_{\alpha,\beta} \in G$. Then for the element $u_\delta v_\mu \in H$, for arbitrary $\delta > 0$ and μ, we have:

$$(u_\delta v_\mu)^{-1}(1 + ae_{\alpha,\beta})(u_\delta v_\mu) = 1 + ae_{\alpha\delta+\mu,\beta\delta+\mu}.$$

Consequently, for $\delta = \dfrac{\lambda - \gamma}{\beta - \alpha} > 0$, $\mu = \dfrac{\beta\gamma - \alpha\lambda}{\beta - \alpha}$, we obtain the required assertion.

Now let $N \subset G$, N be a nonidentity normal divisor of the group H. Then there exists an element $1 + ae_{\gamma,\lambda}$, $\gamma < \lambda$, not belonging to N. In view of Lemma 2, N contains at least one element of the form $1 + ae_{\alpha,\beta}$, $\alpha < \beta$. Then it follows from what was said above that there exist elements u_δ, v_μ such that

$$(u_\delta v_\mu)^{-1}(1 + ae_{\alpha,\beta})(u_\delta v_\mu) = 1 + ae_{\gamma,\lambda}.$$

But this is impossible, since N is invariant in H, $1 + ae_{\alpha,\beta} \in N,\ 1 + ae_{\gamma,\lambda} \notin N$.

Theorem. The group H is not an RI^*-group.

Indeed, suppose H possesses an ascending invariant solvable series:

$$E = H_0 \subset H_1 \subset \cdots \subset H_\alpha \subset H_{\alpha+1} \subset \cdots \subset H_\gamma = H,$$

i.e. all H_α are invariant in H, and $H_{\alpha+1}/H_\alpha$ are abelian. Then G possesses the following ascending invariant solvable series with repetitions, all members of which are invariant in the whole group H:

$$E \subseteq H_1 \cap G \subseteq \cdots \subseteq H_\alpha \cap G \subseteq H_{\alpha+1} \cap G \subseteq \cdots \subseteq H_\gamma \cap G = G.$$

In view of Lemma 4, each member $H_\alpha \cap G$ of this series either coincides with the identity subgroup or coincides with G, but then, by the theorem on subgroups of a group with a normal system (see (4), p. 358), G must be commutative.

§ 3. **Corollary 1.** The class of RI^*-groups is not closed under extensions (answer to a question of Ya. B. Livchak (8)).

Since a radical class is closed under extensions, it follows from this that the class of RI^*-groups is not radical (a negative answer to a question of A. G. Kurosh (1)).

The group H possesses a finite invariant series with RI^*-factors; therefore it is a group of the 3rd degree over the class of abelian groups (see § 1). Consequently, H is a subsolvable group of P. Baer (5). Hence we obtain:

Corollary 2. There exists a subsolvable group that is not an RI^*-group (answer to a question of Baer (5)).

In the paper (9), K. K. Shchukin studied the RI^*-solvable radical R in groups. This radical has the following property: the radical R of a group H is an RI^*-subgroup possessing an ascending solvable series all of whose members are invariant in the group H. All normal divisors of the group H with this property are contained in R (9). For this radical the following question remained open ((9), p. 1029): will the radical R contain every invariant RI^*-subgroup of the group H, i.e., in other words, is it a radical in the sense of the paper (1)?

For the example H (§ 2) it is evident that the radical R of the group H cannot contain the invariant RI^*-subgroup G of this group (see the proof of the theorem), i.e. we have

Corollary 3. The RI^*-radical of K. K. Shchukin (9) is not a radical in the ordinary sense.

In paper (2), for an abstract class of groups Σ, the following condition was introduced:

(β) If a group $A \neq E$, generated by its invariant Σ-subgroups, is a normal divisor of a group B, then A contains a nontrivial Σ-subgroup invariant in the whole group B.

There the question was also posed whether abelian groups satisfy condition (β).

In our example the group G, generated by its abelian invariant subgroups (Lemma 3), is a normal divisor of the group H, but G contains no abelian subgroups invariant in the whole group H. Hence we obtain

Corollary 4. *Condition (β) is not fulfilled for the class of abelian groups.*

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REFERENCES

[1] A. G. Kurosh, DAN, **141**, No. 4 (1961). [2] K. K. Shchukin, DAN, **142**, No. 5 (1962). [3] Chan Van Hao, Sibirsk. matem. zhurn., **4**, No. 6, 969 (1962). [4] A. G. Kurosh, *Theory of Groups*, Moscow, 1953. [5] R. Baer, Math. Zs., **62**, 4, 402 (1955). [6] I. D. Ado, DAN, **40**, No. 8 (1943). [7] D. H. McLain, Proc. Cambr. Phil. Soc., **50**, 4, 641 (1954). [8] Ya. B. Livchak, Sibirsk. matem. zhurn., **1**, No. 4, 617 (1960). [9] K. K. Shchukin, Matem. sborn., **52**, 4, 1021 (1960).

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