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Abstract

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ON UNIVERSAL SPACES FOR CERTAIN CLASSES OF SPACES

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In this note new classes of spaces will be introduced, generalizing the classes of metric spaces and of spaces with a σ-star finite base (^{1}), as well as classes of spaces generalizing the classes of paracompact and strongly paracompact spaces. For these new classes of spaces one can construct universal spaces; in doing so one obtains theorems generalizing, for example, K. Morita' s theorem from (^{2}) and Yu. Nagata' s theorems from (^{1}) (see Corollaries 2 and 3).

I. Definition 1. We shall say that a space $^*X$ has a τ-refining base if in X there is a refining ** system ν of τ covers. If, moreover, all the covers in the system ν are normal ((^{11}), p. 73) and locally finite (respectively, star finite), then we shall say that X has a τ-locally finite (respectively, τ-star finite) base. If, in addition, all covers in the system ν are $(n+1)$-fold, then we shall say that X has a τ-locally (star) finite $(n+1)$-fold base.

Example 1. All completely regular spaces of weight $\leq \tau$ have a τ-star finite base.

Example 2. Metric (n-dimensional in the sense of dim) spaces have an $\aleph_0$-locally finite ($(n+1)$-fold) base.

Example 3. Strongly paracompact (n-dimensional) metric spaces have an $\aleph_0$-star finite ($(n+1)$-fold) base.

Example 4. The product of τ spaces with a τ-refining base again has a τ-refining base.

Example 5. The product of τ spaces with a τ-locally (star) finite base again has a τ-locally (star) finite base.

Example 6. The product X of τ spaces with a single-cover τ-star finite base*** will again be a space with a single-cover τ-star finite base. It is also clear that $\operatorname{ind} X = 0$.

Example 7. The spaces $B(\mathfrak{n}, \tau)$, which are the product of τ spaces N_α, $\alpha \in \mathfrak{A}$, each of which consists of $\mathfrak{n}$ isolated points, have a τ-star finite base.

For $\tau = \aleph_0$ the spaces $B(\mathfrak{n}, \aleph_0)$ coincide with the generalized Baire spaces $B(\mathfrak{n})$ of weight $\mathfrak{n}$. Obviously, $\operatorname{ind} B(\mathfrak{n}, \tau) = 0$. For $\tau > \aleph_0$ the spaces $B(\mathfrak{n}, \tau)$ are completely regular but not normal (3).

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[*] All spaces considered in the note are assumed (unless the contrary is stated) to be completely regular, all covers open, and all mappings continuous.

[**] A system ν of covers ω_α, $\alpha \in \mathfrak{A}$, of a space X is called *refining* if for every point $x \in X$ and every neighborhood Ox of it there is an index $\alpha = \alpha(x, Ox)$ such that $\mathrm{St}(x, \omega_\alpha) \subset Ox$, where $\mathrm{St}(x, \omega_\alpha)$ denotes the union of those elements of the cover ω_α which contain the point x.

[***] We note that a single-cover cover is automatically star finite, and a base decomposing into a system of single-cover covers will automatically be refining. Thus, if a space has a base decomposing into a system of τ single-cover covers, then this space will automatically have a single-cover τ-star finite base.

II. We first consider spaces with a τ-locally finite base, which are a generalization of metric spaces.

Theorem 1. Spaces X of weight κ with a τ-locally finite (and $(n+1)$-fold) base, and only they, are homeomorphic to subsets of products of τ (respectively n-dimensional) metric spaces of weight κ, $\tau \leqslant \kappa$.

Theorem 2. Spaces X of weight κ with a τ-locally finite (and $(n+1)$-fold) base, $\tau \leqslant \kappa$, and only they, are homeomorphic to subsets of products of τ generalized Hilbert spaces H^τ (respectively spaces $F_n(\Omega)$ from (10), universal for n-dimensional metric spaces of weight τ).

Theorem 3. A space X of weight κ that is the product of τ metric spaces ($\tau \leqslant \kappa$) has a resolving mapping $\tilde{f}_X$ with $cw(f_X) \leqslant \kappa$ (4) into the Tikhonov cube I^τ.

Theorem 4. For spaces of weight κ with a τ-locally finite base ($\tau \leqslant \kappa$) there exists a universal bicompactum $X(\kappa, \tau)$ of weight κ, zero-dimensional and openly mapped onto the Tikhonov cube I^τ ($X(\kappa, \tau)$ is a local product over I^τ (5)).

Theorem 4 can be generalized:

Theorem 5. Spaces X that are subsets of products of τ spaces X_α, each of which has a resolving mapping f_α with $cw(f_\alpha) \leqslant \kappa$, $\tau \leqslant \kappa$, onto a completely regular space of weight τ, also have a resolving mapping f with $cw(f) \leqslant \tau$ onto a completely regular space of weight τ; that is, the bicompacta $X(\kappa, \tau)$ indicated in Theorem 4 will be universal for these spaces as well.

III. We turn to spaces with a τ-star finite base. It turns out that metric spaces with a base decomposable into the sum of a countable number of star-finite coverings (in (2) they are called spaces with a σ-star finite base, and in (6)—strongly metrizable spaces) are $\aleph_0$-star finite.

Theorem 6. Completely regular spaces of weight κ with a single τ-star finite base, and only they, are homeomorphic to subsets of $B(\kappa, \tau)$.

For $\tau = \aleph_0$ we obtain the known theorem on the universality of generalized Baire spaces $B(\kappa)$ for zero-dimensional in the sense of dim metric spaces.

Theorem 7. Spaces X of weight κ with a τ-star finite ($(n+1)$-fold) base, $\kappa > \tau$, and only they, are homeomorphic to subsets of products of τ (respectively n-dimensional) metric spaces of weight $\leqslant \kappa$, each of which decomposes into the sum of pairwise disjoint open-closed subsets possessing a countable base.

Theorem 8. Every space of weight κ with a τ-star finite base is homeomorphic to a subset of the product $B(\kappa, \tau) \times I^\tau$.

Remark 1. It is clear that Theorems 7 and 8 strengthen Theorems 1 and 4 (in the case of spaces with a τ-star finite base).

Corollary 1. Every space X of weight κ with a τ-star finite base has such a mapping π into $B(\kappa, \tau)$ that the preimage of each point of $B(\kappa, \tau)$ under this mapping has weight $\leqslant \tau$.

Corollary 2 (2). Every space of weight κ with an $\aleph_0$-star finite base is homeomorphic to a subset of $B(\kappa) \times I^\infty$ (where I^∞ is the Hilbert cube).

Theorem 9. Every space X of weight κ with $\dim X \leqslant n^*$ and with a τ-star finite base is homeomorphic to a subset of the product $F^n \times B(\kappa, \tau)$, where F^n is an n-dimensional (in the sense of dim) bicompactum of weight τ, independent of X.

Corollary 3 (1). Every metric space of weight κ possessing an $\aleph_0$-star finite base is homeomorphic to a subset of: a) the product $B(\kappa) \times M^n$, where M^n is an n-dimensional compactum; b) the product $B(\kappa) \times I^{2n+1}$, where I^{2n+1} is the $(2n+1)$-dimensional cube.

* The dimension dim of a completely regular space is understood in the sense of article (7).

For the proof of the theorems formulated, the following assertion was used:

Theorem 10. *A topological (not necessarily completely regular) space X if and only if, for its (not necessarily open) cover ω, it possesses an ω-mapping* into an n-dimensional metric space (a generalized polyhedron) N, when an $(n+1)$-fold normal locally finite cover γ can be inscribed in the cover ω. If the cover γ is star-finite, then the space N decomposes into a sum of seven pairwise disjoint open-closed subspaces with countable bases.*

This theorem generalizes Dowker' s theorem from ([8]), formulated for normal spaces.

IV. Definition 2. We shall call a space X **τ-normal** if all its covers of cardinality $\leq \tau$ are normal.

Every normal space is ω_0-normal in the sense that any of its finite covers is normal.

It is clear that $\aleph_0$-normality of a space X coincides with its countable paracompactness.

Definition 3. We shall call a cover ν of a space X a **continuation of a system of sets** μ, if every element of the system μ belongs to the cover ν.

We shall call a space X τ**-(strongly) paracompact** if into every cover ω of the space X one can inscribe a cover ω', decomposing into the sum of τ systems, each of which is continued in a locally (star-) finite cover of the space X**.

Lemma. *Let in a τ-normal space X there be given a cover ω, decomposing into the sum of τ systems $\omega_\alpha = \{O_{\alpha\theta}\}$, $\alpha \in \mathfrak{A}$, $\alpha\theta \in \Theta_\alpha$, each of which is continued in a locally (star-) finite cover of the space X. Then each system ω_α can be continued to such a locally (star-) finite cover $\nu_\alpha = \{O_{\alpha\theta}, V_{\alpha\xi}\}$, $\alpha\theta \in \Theta_\alpha$, $\alpha\xi \in \Xi_\alpha$, that for every point $x \in X$ there is an index α for which*

$$x \in \bigcup_{\alpha\theta} O_{\alpha\theta} \setminus \bigcup_{\alpha\xi} V_{\alpha\xi} = X \setminus \bigcup_{\alpha\xi} V_{\alpha\xi}.$$

From Lemma 1 it follows:

Theorem 11. *If a τ-normal and τ-(strongly) paracompact space X has a refining system of τ covers, then the space X has a τ-locally (star-) finite base.*

Besides Theorem 11, Lemma 1 also implies:

Theorem 12. *If in a τ-normal space X there is a cover ω, decomposing into the sum of τ systems ω_α, each of which is continued: a) to a locally finite, b) to a star-finite cover, then the space X has an ω-mapping into the product of τ spaces: a) metric spaces, b) metric spaces decomposing into a sum of pairwise disjoint open-closed subspaces with countable bases.*

In particular, the following is true:

Theorem 13. *An $\aleph_0$-strongly paracompact (completely paracompact in the terminology of note $(^6)$) space X, for any of its covers ω, has an ω-mapping into the product of a countable number of metric spaces, each of which decomposes into a sum of pairwise disjoint open-closed subspaces with a countable base, i.e. X, for any of its covers ω, has an ω-mapping into $I^\infty \times B(\varkappa)$, where $\varkappa$ is the weight of X. If $\dim X = n$, then in this case X, for any of its covers ω, has an ω-mapping into the product $M^n \times B(\varkappa)$, where M^n is an n-dimensional compactum (not depending on X).*

* A mapping $f : X \to R$ is called an ω-mapping for the cover ω if for every point $y \in R$ there exists a neighborhood Oy such that the set $f^{-1}(Oy)$ is contained in one of the elements of the cover ω.

** Obviously, completely paracompact spaces $(^6)$ coincide with $\aleph_0$-strongly paracompact spaces.

Thus, an $\aleph_0$-strongly paracompact space, for each of its covers ω, has an ω-mapping onto a space with an $\aleph_0$-star finite base.

From Theorems 10, 13 and from Theorem 6 of paper (9) it follows:

Theorem 14. a) Strongly paracompact spaces X (with $\dim X = n$) are spectrally decomposable (9), with respect to the class of n-dimensional strongly paracompact metric spaces, each of which is represented in the form of a sum of pairwise disjoint open-and-closed subspaces with a countable base.

b) $\aleph_0$-strongly paracompact spaces X (with $\dim X = n$) are spectrally decomposable with respect to the class of n-dimensional metric spaces with an $\aleph_0$-star finite base.

Analogous assertions can be formulated for τ-normal τ-(strongly) paracompact spaces (see Theorem 12).

Theorem 1 of (9) can be supplemented by the following theorem:

Theorem 15. A mapping $f : X \to R$ of an n-dimensional space X, in the sense of dim: a) onto a strongly paracompact metric space; b) onto a space decomposing into a sum of pairwise disjoint open-and-closed subspaces with a countable base; c) onto a metric space with an $\aleph_0$-star finite base, can be represented as the superposition of two mappings $g : X \to S$ and $h : S \to R$, where S is a metric space of weight equal to the weight of R, n-dimensional in the sense of dim, and respectively: a) strongly paracompact, b) decomposing into a sum of pairwise disjoint open-and-closed subspaces with a countable base, c) possessing an $\aleph_0$-star finite base (and the set $g(X)$ is everywhere dense in S).

Let us also note that the space X will be τ-(strongly) paracompact if it has a closed and bicompact mapping onto a τ-(strongly) paracompact space; in particular, the space X is $\aleph_0$-strongly paracompact (i.e. fully paracompact) if it has a bicompact and closed mapping onto an $\aleph_0$-strongly paracompact space.

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