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Yu. I. SMIRNOV

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Abstract

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MATHEMATICS

Yu. I. SMIRNOV

COMPUTATION OF THE MATHEMATICAL EXPECTATION OF A QUASI-ADDITIVE FUNCTION OF A PATH ON A GRAPH

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We shall consider a multigraph G satisfying conditions (1) and (2) of paper ([1]). Let all inputs of each vertex α of the graph G be numbered by the numbers from 1 to i_α, and all outputs by the numbers from 1 to k_α. By ${}^i a_k$ we shall denote the pair: the i-th input and the k-th output of the vertex α. A sequence

$$ {}^i s_{kn} = {}^i\alpha_{k_1}^{j_1}\, {}^{i_2}\alpha_{k_2}^{j_2} \dots {}^{i_n}\alpha_{k_n}^{j_n} $$

such that, for $l = 1, 2, \dots, n-1$, $(\alpha_{k_l}^{j_l}\, {}^{i_{l+1}}\alpha^{j_{l+1}})$ is an edge of the graph G, will be called a **path** of the graph G. If

$$ {}^i s_k^1 = {}^i\alpha_{k_1}^{j_1}\, {}^{i_2}\alpha_{k_2}^{j_2} \dots {}^{i_n}\alpha_k^{j_n}, \qquad {}^l s_m^2 = {}^l\alpha_{m_1}^{q_1}\, {}^{l_2}\alpha_{m_2}^{q_2} \dots {}^{l_2}\alpha_m^{q_2} $$

and $(\alpha_k^{j_n}\, {}^l\alpha^{q_1})$ is an edge of the graph G, then we shall say that the path

$$ {}^i s_m = {}^i\alpha_{k_1}^{j_1}\, {}^{i_2}\alpha_{k_2}^{j_2} \dots {}^{i_n}\alpha_n^{j_n}\, {}^l\alpha_{m_1}^{q_1}\, {}^{l_2}\alpha_{m_2}^{q_2} \dots {}^{l_2}\alpha_m^{q_2} $$

is composed of the paths ${}^i s_k^1$ and ${}^l s_m^2$, and we shall write ${}^i s_m = {}^i s_k^1\, {}^l s_m^2$. A function $f(s)$ of a path of the graph will be called **quasi-additive** if there exists a path function $\varphi(s)$ such that, for any s^1 and s^2 such that $s = s^1 s^2$, the conditions

$$ \varphi(s) = \varphi(s^1) \cdot \varphi(s^2); \tag{1} $$

$$ f(s) = f(s^1) + \varphi(s^1) f(s^2). \tag{2} $$

are satisfied.

Fig. 1

Fig. 1

Figure 1: Fig. 1

Let, for each pair ${}^{i}a_{k}$, a number $p({}^{i}a_{k}) \geq 0$ be given—the probability of exiting through the k-th output of the vertex α under the condition that one entered through the i-th input of the vertex α. Obviously,

$$\sum_{k=1}^{k_\alpha} p({}^{i}a_{k}) = 1.$$

Define the probability $p({}^{i_1}s_{kn})$ of the path

$${}^{i_1}s_{kn} = {}^{i_1}\alpha_{k_1}^{j_1}\, {}^{i_2}\alpha_{k_2}^{j_2} \ldots {}^{i_n}\alpha_{k_n}^{j_n},$$

by putting

$$p({}^{i_1}s_{kn}) = \prod_{t=1}^{n} p({}^{i_t}\alpha_{k_t}^{j_t}).$$

Let the graph G have an input and an output (see ([1])). We set ourselves the task of computing

$$Mf(\bar{s}) = \sum_{\bar{s}} p(\bar{s}) f(\bar{s}).$$

Here $\bar{s}$, as in paper ([2]), denotes a path going from the input to the output. Let ${}^{i}\alpha_{k_1}$ be a loop at the vertex α. Consider the paths

$${}^{i}_{-1}s_k = {}^{i}\alpha_k, \qquad {}^{i}_{0}s_k = {}^{i}\alpha_{k_1}\, {}^{i_1}\alpha_k, \qquad {}^{i}_{1}s_k = {}^{i}\alpha_{k_1}\, {}^{i_1}\alpha_{k_1}\, {}^{i_1}\alpha_k, \qquad {}^{i}_{2}s_k = {}^{i}\alpha_{k_1}\, {}^{i_1}\alpha_{k_1}\, {}^{i_1}\alpha_{k_1}\, {}^{i_1}\alpha_k, \ldots$$

$$\ldots, \quad {}^{i}_{n}s_k = {}^{i}\alpha_{k_1} \underbrace{{}^{i_1}\alpha_{k_1}\, {}^{i_1}\alpha_{k_1} \ldots {}^{i_1}\alpha_{k_1}}_{n}\, {}^{i_1}\alpha_k.$$

Compute $\varphi({}^{i}_{n}s_k)$. From (1) we have

$$\varphi({}^{i}_{n}s_k) = \varphi({}^{i}\alpha_{k_1})\, \varphi^{n}({}^{i_1}\alpha_{k_1})\, \varphi({}^{i_1}\alpha_k) \quad \text{for } n \geq 0. \tag{3}$$

Compute $f({}^{i}_{n}s_k)$. Introduce the notation

$$ {}^{i}_{n}\bar{s}_k = {}^{i}\alpha_{k_1}\,\underbrace{{}^{i_1}\alpha_{k_1}\ldots{}^{i_1}\alpha_{k_1}}_{n}, \qquad n \geq 0. $$

Obviously,

$$ f({}^{i}_{0}\bar{s}_k) = f({}^{i}\alpha_{k_1}), \qquad f({}^{i}_{1}\bar{s}_k) = f({}^{i}\alpha_{k_1}) + \varphi({}^{i}\alpha_{k_1}) f({}^{i_1}\alpha_{k_1}). $$

From (2) and (3),

$$ f({}^{i}_{n}\bar{s}_{k_1}) = f({}^{i}\alpha_{k_1}) + \varphi({}^{i}\alpha_{k_1}) \left[\sum_{t=1}^{n} \varphi^{t-1}({}^{i_1}\alpha_{k_1})\right] f({}^{i_1}\alpha_{k_1}) \quad \text{for } n \geq 1. $$

Since ${}^{i}_{n}s_k = {}^{i}_{n}s_{k_1}\,{}^{i}\alpha_k$, from the quasiadditivity of $f(s)$ we obtain

$$ f({}^{i}_{n}s_k) = f({}^{i}\alpha_{k_1}) + \varphi({}^{i}\alpha_{k_1}) \left[\sum_{t=1}^{n} \varphi^{t-1}({}^{i_1}\alpha_{k_1})\right] f({}^{i_1}\alpha_{k_1}) + \varphi({}^{i}\alpha_{k_1}) \varphi^{n}({}^{i_1}\alpha_{k_1}) f({}^{i_1}\alpha_k). \tag{4} $$

If the element $E - \varphi({}^{i_1}\alpha_{k_1})$ has an inverse, then

$$ f({}^{i}_{n}s_k) = f({}^{i}\alpha_{k_1}) + \varphi({}^{i}\alpha_{k_1}) \frac{E - \varphi^{n}({}^{i_1}\alpha_{k_1})}{E - \varphi({}^{i_1}\alpha_{k_1})} f({}^{i_1}\alpha_{k_1}) + \varphi({}^{i}\alpha_{k_1}) \varphi^{n}({}^{i_1}\alpha_{k_1}) f({}^{i_1}\alpha_k). $$

Lemma.

$$ \sum_{n=-1}^{\infty} p({}^{l}s^{1}_{m}\,{}^{i}_{n}s_k\,{}^{p}s^{3}_{q})\, f({}^{l}s^{1}_{m}\,{}^{i}_{n}s_k\,{}^{p}s^{3}_{q}) = p({}^{l}s^{1}_{m}\,{}^{i}\tilde{s}_k\,{}^{p}s^{3}_{q})\, f({}^{l}s^{1}_{m}\,{}^{i}\tilde{s}_k\,{}^{p}s^{3}_{q}), $$

where

$$ p({}^{i}\tilde{s}_k) = p({}^{i}\alpha_k) + \sum_{n=0}^{\infty} p({}^{i}\alpha_{k_1}) p^{n}({}^{i_1}\alpha_{k_1}) p({}^{i_1}\alpha_k), $$

$$ \varphi({}^{i}\tilde{s}_k) = \frac{\sum\limits_{n=-1}^{\infty} p({}^{i}_{n}s_k)\varphi({}^{i}_{n}s_k)}{p({}^{i}\tilde{s}_k)}, \qquad f({}^{i}\tilde{s}_k) = \frac{\sum\limits_{n=-1}^{\infty} p({}^{i}_{n}s_k) f({}^{i}_{n}s_k)}{p({}^{i}\tilde{s}_k)}. $$

Proof. By virtue of the definition of $p(s)$ and the quasiadditivity of $f(s)$, we have

$$\sum_{n=-1}^{\infty} p({}^l s^1_m\, {}^i_n s_k\, {}^p s^3_q) f({}^l s^1_m\, {}^i_n s_k\, {}^p s^3_q) =$$

$$= \sum_{n=-1}^{\infty} p({}^l s^1_m) p({}^i_n s_k) p({}^p s^3_q) \left[f({}^l s^1_m\, {}^i_n s_k) + \varphi({}^l s^1_m\, {}^i_n s_k) f({}^p s^3_q) \right] =$$

$$= p({}^l s^1_m) p({}^p s^3_q) \sum_{n=-1}^{\infty} p({}^i_n s_k) \left[f({}^l s^1_m) + \varphi({}^l s^1_m) f({}^i_n s_k) + \varphi({}^l s^1_m) \varphi({}^i_n s_n) f({}^p s^3_q) \right] =$$

$$= p({}^l s^1_m) p({}^i \tilde{s}_k) p({}^p s^3_q) \left[f({}^l s^1_m) + \varphi({}^l s^1_m) \frac{\sum_{n=-1}^{\infty} p({}^i_n s_k) f({}^i_n s_k)}{p({}^i \tilde{s}_k)} + \right.$$

$$\left. + \varphi({}^l s^1_m) \frac{\sum_{n=-1}^{\infty} p({}^i_n s_k) \varphi({}^i_n s_k)}{p({}^i \tilde{s}_k)} f({}^p s^3_q) \right] =$$

$$= p({}^l s^1_m\, {}^i \tilde{s}_k\, {}^p s^3_q) \left[f({}^l s^1_m) + \varphi({}^l s^1_m) f({}^i \tilde{s}_k) + \varphi({}^l s^1_m) \varphi({}^i \tilde{s}_k) f({}^p s^3_q) \right] =$$

$$= p({}^l s^1_m\, {}^i \tilde{s}_k\, {}^p s^3_q) f({}^l s^1_m\, {}^i \tilde{s}_k\, {}^p s^3_q),$$

which was required to prove.

Remark.

$$p({}^i \tilde{s}_k) = p({}^i \alpha_k) + \frac{p({}^i \alpha_{k_1}) p({}^{i_1} \alpha_k)}{1 - p({}^{i_1} \alpha_{k_1})}.$$

If the element $E - p({}^{i_1} \alpha_{k_1}) \varphi({}^{i_1} \alpha_{k_1})$ has an inverse, then

$$\varphi({}^i \tilde{s}_k) = \frac{p({}^i \alpha_k) \varphi({}^i \alpha_k) + p({}^i \alpha_{k_1}) p({}^{i_1} \alpha_k) \varphi({}^i \alpha_{k_1}) \left(E - p({}^{i_1} \alpha_{k_1}) \varphi({}^{i_1} \alpha_{k_1}) \right)^{-1} \varphi({}^{i_1} \alpha_k)}{p({}^i \tilde{s}_k)}.$$

If the element $E - \varphi({}^{i_1} \alpha_{k_1})$ also has an inverse, then

$$f({}^i\tilde{s}_k) = \Bigg[p({}^i\alpha_k)f({}^i\alpha_k) + p({}^i\alpha_{k_1})p({}^i\alpha_k)\Bigg\{\frac{f({}^i\alpha_{k_1})}{1-p({}^i\alpha_{k_1})} + \\ + \varphi({}^i\alpha_{k_1})\frac{E}{E-p({}^i\alpha_{k_1})\varphi({}^i\alpha_{k_1})} f({}^i\alpha_k) + \frac{2p({}^{i_1}\alpha_{k_1})}{1-p({}^{i_1}\alpha_{k_1})}\varphi({}^i\alpha_{k_1})f({}^i\alpha_{k_1}) - \\ - \varphi({}^i\alpha_{k_1})\left(E-\varphi({}^{i_1}\alpha_{k_1})\right)^{-1}\left(E-p({}^{i_1}\alpha_{k_1})\varphi({}^{i_1}\alpha_{k_1})\right)^{-1} f({}^{i_1}\alpha_{k_1})\Bigg\}\Bigg]/p({}^i\tilde{s}_k).$$

Transformation 1 of the graph G. Construct, starting from the graph G, a graph G' as follows: in the graph G remove the loop ${}^{i_1}\alpha_{k_1}$. We shall mark all elements of the graph G' and the functions specified on them by a prime. Put

$$p'({}^i\beta'_k) = \begin{cases} p({}^i\beta_k), & \text{if } \beta \neq \alpha, \\ p({}^i\tilde{s}_k), & \text{if } \beta = \alpha; \end{cases} \qquad \varphi'({}^i\beta'_k) = \begin{cases} \varphi({}^i\beta_k), & \text{if } \beta \neq \alpha, \\ \varphi({}^i\tilde{s}_k), & \text{if } \beta = \alpha; \end{cases}$$

$$f'({}^i\beta'_k) = \begin{cases} f({}^i\beta_k), & \text{if } \beta \neq \alpha, \\ f({}^i\tilde{s}_k), & \text{if } \beta = \alpha. \end{cases}$$

A path ${}^{q_1}s'_{\nu_t}$ of the graph G' can be represented uniquely in the form

$${}^{q_1}s'_{\nu_t} = {}^{q_1}s^1_{\nu_1}\,{}^{m_1}\alpha'_{k_1}\,{}^{q_2}s^2_{\nu_2}\,{}^{m_2}\alpha'_{k_2} \dots {}^{q_{t-1}}s^{t-1}_{\nu_{t-1}}\,{}^{m_{t-1}}\alpha'_{k_{t-1}}\,{}^{q_t}s^t_{\nu_t}$$

so that the paths ${}^{q_j}s^j_{\nu_j}$ $(j = 1, 2, \dots, t)$ do not contain the vertex α' (in this case some of the paths ${}^{q_j}s^j_{\nu_j}$ may be empty). To the path ${}^{q_1}s'_{\nu_t}$ of the graph G' let us put in correspondence all paths of the graph G having the form

$${}^{q_1}s^1_{\nu_1}\,{}^{m_1}_{n_1}s_{k_1}\,{}^{q_2}s^2_{\nu_2} \dots {}^{m_{t-1}}_{n_{t-1}}s_{k_{t-1}}\,{}^{q_t}s^t_{\nu_t},$$

$$j = 1, 2, \dots, t-1; \qquad n_j = -1, 0, 1, 2, \dots .$$

We denote this set of paths by $s({}^{q_1}s'_{\nu_t})$. We shall compute

$$\sum_{s({}^{q_1}s'_{\nu_t})} p(s)f(s) =$$

$$= \sum_{j=1}^{t-1}\sum_{n_j=-1}^{\infty} p({}^{q_1}s^1_{\nu_1}\,{}^{m_1}_{n_1}s_{k_1}\,{}^{q_2}s^2_{\nu_2} \dots {}^{m_{t-1}}_{n_{t-1}}s_{k_{t-1}}\,{}^{q_t}s^t_{\nu_t})f({}^{q_1}s^1_{\nu_1}\,{}^{m_1}_{n_1}s_{k_1}\,{}^{q_2}s^2_{\nu_2} \dots {}^{m_{t-1}}_{n_{t-1}}s_{k_{t-1}}\,{}^{q_t}s^t_{\nu_t}).$$

Theorem.

$$\sum_{s({}^{q_1}s'_{\nu_t})} p(s)f(s) = p({}^{q_1}s'_{\nu_t})f({}^{q_1}s'_{\nu_t}).$$

Proof. By the lemma,

$$\sum_{n_1=-1}^{\infty} p({}^{q_1}s^1_{\nu_1}\,{}^{m_1}_{n_1}s_{k_1} \ldots {}^{q_t}s^t_{\nu_t})f({}^{q_1}s^1_{\nu_1}\,{}^{m_1}_{n_1}s_{k_1} \ldots {}^{q_t}s^t_{\nu_t}) =$$

$$= p({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2}\,{}^{m_2}_{n_2}s_{k_2} \ldots {}^{q_t}s^t_{\nu_t})f({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2} \ldots {}^{q_t}s^t_{\nu_t}).$$

Suppose

$$\sum_{s({}^{q_1}s'_{\nu_t})} p(s)f(s) = \sum_{j=r}^{t-1}\sum_{n_j=-1}^{\infty} p({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2}\,{}^{m_2}\tilde{s}_{k_2} \ldots$$

$$\ldots {}^{q_{r-1}}s^{r-1}_{\nu_{r-1}}\,{}^{m_{r-1}}\tilde{s}_{k_{r-1}}\,{}^{q_r}s^r_{\nu_r}\,{}^{m_r}_{n_r}s_{k_r}\,{}^{q_{r+1}}s^{r+1}_{\nu_{r+1}} \ldots {}^{q_t}s^t_{\nu_t})f({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2}\,{}^{m_2}\tilde{s}_{k_2} \ldots$$

$$\ldots {}^{q_{r-1}}s^{r-1}_{\nu_{r-1}}\,{}^{m_{r-1}}\tilde{s}_{k_{r-1}}\,{}^{q_r}s^r_{\nu_r}\,{}^{m_r}_{n_r}s_{k_r}\,{}^{q_{r+1}}s^{r+1}_{\nu_{r+1}} \ldots {}^{q_t}s^t_{\nu_t}).$$

Since, by the lemma,

$$\sum_{n_r=-1}^{\infty} p\left({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2} \ldots {}^{m_{r-1}}\tilde{s}_{k_{r-1}}\,{}^{q_r}s^r_{\nu_r}\,{}^{m_r}_{n_r}s_{k_r} \ldots {}^{q_t}s^t_{\nu_t}\right) f\left({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2} \ldots {}^{m_{r-1}}\tilde{s}_{k_{r-1}}\,{}^{q_r}s^r_{\nu_r}\,{}^{m_r}_{n_r}s_{k_r} \ldots {}^{q_t}s^t_{\nu_t}\right)$$

$$= p\left({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2} \ldots {}^{q_r}s^r_{\nu_r}\,{}^{m_r}\tilde{s}_{k_r}\,{}^{q_{r+1}}s^{r+1}_{\nu_{r+1}}\,{}^{m_{r+1}}_{n_{r+1}}s_{k_{r+1}} \ldots {}^{q_t}s^t_{\nu_t}\right)$$

$$\ldots {}^{q_t}s^t_{\nu_t}\; f\left({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1}\,{}^{q_2}s^2_{\nu_2} \ldots {}^{q_r}s^r_{\nu_r}\,{}^{m_r}\tilde{s}_{k_r}\,{}^{q_{r+1}}s^{r+1}_{\nu_{r+1}}\,{}^{m_{r+1}}_{n_{r+1}}s_{k_{r+1}} \ldots {}^{q_t}s^t_{\nu_t}\right),$$

we have

$$\sum_{s({}^{q_1}s^t_{\nu_t})} p(s)f(s) = \sum_{j=r+1}^{t-1}\sum_{n_j=-1}^{\infty} p\left({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1} \ldots {}^{m_r}\tilde{s}_{k_r}\,{}^{q_{r+1}}s^{r+1}_{\nu_{r+1}}\,{}^{m_{r+1}}_{n_{r+1}}s_{k_{r+1}} \ldots\right)$$

$$\ldots {}^{q_t}s^t_{\nu_t} f\left({}^{q_1}s^1_{\nu_1}\,{}^{m_1}\tilde{s}_{k_1} \ldots {}^{m_r}\tilde{s}_{k_r}\,{}^{q_{r+1}}s^{r+1}_{\nu_{r+1}}\,{}^{m_{r+1}}_{n_{r+1}}s_{k_{r+1}} \ldots {}^{q_t}s^t_{\nu_t}\right).$$

Fig. 2

Figure 2: Fig. 2

Hence, by induction, we obtain

$$\sum_{s({}^{q_1}s^t_{\nu_t})} p(s)f(s) = p\left({}^{q_1}s^t_{\nu_t}\right) f\left({}^{q_1}s^t_{\nu_t}\right),$$

which was required to be proved.

It follows from the theorem that transformation 1 of the graph does not change $Mf(\bar{s})$.

Transformation 2 of the graph G. Let a^1 and a^2 be adjacent vertices of the graph G; $l^1_1, l^1_2, \dots, l^1_{n_1}$ are the inputs of vertex a^1; $l^2_1, l^2_2, \dots, l^2_{n_2}$ are the inputs of vertex a^2; $k^1_1, k^1_2, \dots, k^1_{m_1}$ are the outputs of vertex a^1; $k^2_1, k^2_2, \dots, k^2_{m_2}$ are the outputs of vertex a^2; $k^1_1, k^1_2, \dots, k^1_{i_1}$ are those outputs of a^1 which lead to the inputs of vertex a^2; $k^2_1, k^2_2, \dots, k^2_{i_2}$ are those outputs of a^2 which lead to the inputs of a^1.

Fig. 2

It is easy to see that, in computing $Mf(\bar{s})$, the pair of vertices a^1 and a^2 may be replaced by a vertex a' with inputs $l^1_1, l^1_2, \dots, l^1_{n_1}, l^2_1, l^2_2, \dots, l^2_{n_2}$ and outputs $k^1_1, k^1_2, \dots, k^1_{m_1}, k^2_1, k^2_2, \dots, k^2_{m_2}$, putting

$$p\left({}^k a'_l\right) = \begin{cases} p\left({}^{k^t_m} a^{t_\nu}_{l^t_q}\right), & \text{if } k = k^t_m,\ l = l^\nu_q \text{ and } t = \nu, \\ 0, & \text{otherwise;} \end{cases}$$

$$\varphi\left({}^k a'_l\right) = \begin{cases} \varphi\left({}^{k^t_m} a^{t_\nu}_{l^t_q}\right), & \text{if } k = k^t_m,\ l = l^\nu_q \text{ and } t = \nu, \\ 0, & \text{otherwise.} \end{cases}$$

$$f\left({}^k a'_l\right) = \begin{cases} f\left({}^{k^t_m} a^{t_\nu}_{l^t_q}\right), & \text{if } k = k^t_m,\ l = l^\nu_q \text{ and } t = \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Transformation 2, without changing $Mf(\bar{s})$, reduces the number of vertices of the graph; transformation 1, without changing $Mf(\bar{s})$, removes loops at vertices of the graph. Thus, by these transformations the graph can be reduced to a single vertex $\bar{a}$ with one input k_1 and one output l_1. Then $Mf(\bar{s}) = f({}^{k_1}a_{l_1})$.

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Note: Figure translations are in progress. See original paper for figures.