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Abstract

Full Text

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ON NORMAL DIVISORS OF THE MULTIPLICATIVE GROUP OF A DIVISION RING

(Presented by Academician P. S. Novikov on 18 IV 1963)

The Cartan–Brauer–Hua theorem is known (see, for example, ⁽¹⁾, p. 271):

The only subfields of a division ring Δ that are invariant under all inner automorphisms are Δ and the subfields of the center Φ of the division ring Δ .

This theorem gives some information about normal divisors of the multiplicative group of a division ring.

In Scott's paper ⁽²⁾ the following result was obtained:

The factor group of the multiplicative group of a division ring by its center contains no abelian normal divisors.

A number of recent works have been devoted to generalizations of the Cartan–Brauer–Hua and Scott theorems, in particular those of M. Sh. Khuzurbazar ^(6, 7).

From Jacobson's theorem (⁽¹⁾, p. 314) it follows that the multiplicative group of a noncommutative division ring cannot be periodic. It may therefore be conjectured that all periodic normal divisors of the multiplicative group of a division ring are contained in the center. For division rings locally finite over their center, this will follow from Theorem 1 of the present paper; in the general case the question remains open. In the present paper, however, it is proved that the factor group of the multiplicative group of a division ring by its center contains no locally finite normal divisors.

In what follows, D denotes an arbitrary division ring, Z its center, and D^* and Z^* the multiplicative groups of D and Z , respectively.

Lemma 1. *Let D be a division ring of characteristic $p \neq 0$, and let F be a locally finite normal divisor of the group D^* . Then $F \subset Z^*$.*

Proof. As noted in ⁽³⁾, Theorem 6, every finite subgroup of a division ring of prime characteristic is cyclic. Consequently, F is an abelian group. From Scott's theorem ⁽²⁾ it follows that $F \subset Z^*$. The lemma is proved.

Lemma 2. *Let D be a division algebra over the field R of rational numbers, with dimension $(D : R) = n$. Let ξ_1, \dots, ξ_l be roots of unity in the division ring D ; let α_i be rational integers; and let β be a rational number whose denominator is relatively prime to n . If the equality*

$$\sum \alpha_i \xi_i = \beta, \quad (1)$$

holds, then β is a rational integer.

Proof. Choose in the algebra D a basis $e_1 = 1, e_2, \dots, e_n$. To each element $b \in D$ we put in correspondence a linear transformation B according to the following rule: $xB = xb$ for all x in D . Then we obtain an isomorphic representation of the algebra D in the matrix algebra R_n . From equality (1) we obtain

$$\sum \alpha_i B_i = \beta E_n, \quad (2)$$

where B_i is the matrix corresponding to ξ_i in our representation. Note that for each matrix B_i there exists a number $n_i \geq 1$ such that $B_i^{n_i} = E_n$. Therefore all characteristic roots of the matrix B_i are roots of unity. Consequently, $\text{Sp}(B_i) = q_i$, where q_i is an algebraic integer. From equality (2) it follows that

$$\sum \alpha_i \text{Sp}(B_i) = n\beta. \quad (3)$$

In equality (3), on the left there is an algebraic integer. Since a rational number is an algebraic integer if and only if it is a rational integer, $n\beta$ is a rational integer. From the condition of the lemma it now follows that β is a rational integer.

Lemma 3. *Let D be a division algebra of dimension n over the field R of rational numbers; let F be a periodic normal subgroup of the group D^* ; let p be a prime number not dividing n ; and let C_p be the ring of p -integral rational numbers. Consider the subring A_p of the division ring D consisting of elements $x = \sum \alpha_i g_i$, where $\alpha_i \in C_p$, $g_i \in F$. Then A_p does not contain the element p^{-1} .*

Proof. Suppose that $\sum \alpha_i g_i = p^{-1}$. Then, reducing the α_i to a common denominator s , we obtain $\sum \beta_i g_i = sp^{-1}$, where sp^{-1} is an irreducible fraction, and β_i and s are rational integers. From Lemma 2 it now follows that sp^{-1} is a rational integer, which is impossible. The contradiction obtained proves the lemma.

Lemma 4. *Suppose that the conditions of Lemma 3 are satisfied. Then in the ring A_p all elements $y = \sum \gamma_i g_i$, where $\gamma_i \equiv 0 \pmod{p}$ in the ring C_p , form a proper ideal T . In the factor ring by this ideal all nilpotent elements form a nilpotent ideal.*

Proof. If the ideal T coincided with the ring A_p , then the equality $1 = \sum \gamma_i g_i$ would hold, where $\gamma_i \equiv 0 \pmod{p}$ in the ring C_p . It would follow that the element p is invertible in the ring A_p , which is impossible in view of Lemma 3. Consider now the quotient ring of the ring A_p by the ideal T . It is clear that the ring $\mathcal{L} \cong A_p/T$ is a module over the ring C_p , and the ideal p of the ring C_p annihilates the ring \mathcal{L} . Therefore \mathcal{L} is a module over the quotient ring $C_p/(p)$,

which we shall denote by Ω . Since in the ring A_p any $n+1$ elements are linearly dependent over C_p , and Ω is contained in the center of \mathcal{L} , it follows that \mathcal{L} is a finite-dimensional algebra over the field Ω of p elements.

We now prove that for any elements v, u of the ring A_p the equalities

$$(uv)^m = u_1 v^m, \quad (vu)^m = v^m u_2, \quad (4)$$

hold, where m is an arbitrary positive integer, $u_1, u_2 \in A_p$. Indeed, the ring A_p is invariant with respect to all inner automorphisms of the division ring D . Suppose that for all $k < m$ (4) has already been proved. We may assume that $v \neq 0$. Then

$$\begin{aligned} (uv)^m &= (uv)(uv)^{m-1} = (uv)u'v^{m-1} = u(vu'v^{-1})v^m = \\ &= uu''v^m = u_1 v^m, \quad \text{where } u_1 = uu'', \quad u_1 \in A_p. \end{aligned}$$

The second equality is proved in exactly the same way.

It follows from (4) that every nilpotent element in the ring \mathcal{L} generates a nilpotent ideal. Consequently, the Jacobson radical of the ring \mathcal{L} coincides with the set of all nilpotent elements.

Lemma 5. *Let D be a division algebra, finite-dimensional over the field of rational numbers, and let F be a periodic normal subgroup of the group D^* . Then $F \subset Z^*$.*

Proof. Consider the preimage of the radical of the ring \mathcal{L} (see the proof of Lemma 4) under the natural homomorphism $A_p \rightarrow \mathcal{L}$. Denote this ideal by T^* . The ideal $T^* \supseteq T$, and the quotient ring $Q \cong A_p/T^*$ contains no nilpotent elements. Moreover, the algebra Q is finite-dimensional over the field Ω . From (1), p. 315, it follows that the algebra Q is commutative. Therefore the commutator ideal K of the algebra A_p is contained in T^* . There exists a positive integer r such that $(T^*)^r \subset T$. In particular, $(g_i g_j - g_j g_i)^r \subset T$, $g_i, g_j \in F$. Hence we obtain that the element $g_i g_j - g_j g_i$ is not invertible in the ring A_p . Now denote the element $g_j^{-1} g_i^{-1} g_j g_i$ by q . The element $1 - q$ is not invertible in the ring A_p . We shall show that it follows from this that $q = 1$.

Indeed, otherwise there exists a polynomial irreducible over R , $\varphi(x) = x^r + \lambda_1 x^{r-1} + \dots + \lambda_r$ with integral coefficients, such that the element $1 - q$ is its root. Recall that the element q is a root of unity, and therefore $1 - q$ is an integral algebraic number. Substituting the element $1 - q$ into the polynomial $\varphi(x)$, we obtain

$$(1 - q)\psi(1 - q) = -\lambda_r, \quad (5)$$

where $\psi(x)$ is a polynomial with integral rational coefficients.

Recall now that the number p was subject to the sole condition $(p, n) = 1$, where n is the degree of D over R . We now require also that p satisfy the condition

$(p, \lambda_r) = 1$. Then it follows from (5) that the element $1 - q$ is invertible in the ring A_p . This contradiction shows that $(1 - q) = 0$. Since the elements g_i, g_j were arbitrary, the group F is abelian. It now follows from Scott's theorem that $F \subset Z^*$.

Lemma 6. *Let D be an arbitrary division ring, and let F be a locally finite normal divisor of the group D^* . Then $F \subset Z^*$.*

Proof. In view of Lemma 1, we may assume that D is a division ring of characteristic zero. Consider a finite subgroup F_1 of the group F , generated by the elements $g_1 = 1, g_2, \dots, g_s$. The ring P , consisting of elements $\tau = \sum \alpha_i f_i$, where $\alpha_i \in R$, $f_i \in F$, is a division ring, since P is a finite-dimensional algebra over R without zero divisors. The group P^* contains the normal divisor $P^* \cap F$. In view of Lemma 5, $P^* \cap F$ is an abelian group. Hence $F_1 \subset P^* \cap F$ is also an abelian group. Since the elements $g_1 = 1, g_2, \dots, g_s$ are arbitrary, the group F is abelian. This proves the lemma.

Theorem 1. *Let D be a division ring locally finite over its center Z . Then the factor group D^*/Z^* contains no nontrivial periodic normal divisors.*

Proof. Suppose first that the division ring D is finite-dimensional over its center. Let G be a normal divisor in D^* , each element g of which satisfies the condition $g^k = z_g$, where k is some positive number depending on g , and $z_g \in Z^*$.

Consider the representation of the group G , as was done in the proof of the lemma. Recall that to an element g there corresponds the linear transformation B_g according to the following law: $xg = xB_g$, where x is an arbitrary element of D . In this case an element z of $G \cap Z^*$ passes into the matrix zE_n , where n is the dimension of D over Z . From the equality $g^k = z_g$ it follows that

$$(B_g)^k = z_g E_n. \quad (6)$$

From (6) we conclude that $B_g^{kk_1} = E_n$ if and only if $z_g^{k_1} = 1$, where k_1 is any positive number. But $(\det B_g)^k = z_g^n$. Therefore $\det B_g$ is a root of unity if and only if z_g is a root of unity. Hence $(B_g)^t = E_n$ for some positive t if and only if $\det B_g$ is a root of unity. It follows from this, by virtue of the faithfulness of the representation, that an element g has finite order if and only if $\det B_g$ is a root of unity. Consequently, all elements of finite order in the group G form a subgroup F , which is a normal divisor in D^* . But the group F is isomorphic to a matrix group. From (4) we obtain that F is locally finite. Lemma 6 now permits us to conclude that $F \subset Z^*$. Note that if an element of the group G is a commutator of elements f_1 and f_2 , i.e. $f = f_1^{-1} f_2^{-1} f_1 f_2$, then B_f is a matrix with determinant equal to 1. From the preceding considerations it is clear that the commutant of the group G is contained in F , and hence in Z^* . Thus the group G is metabelian. From (2) it follows that $G \subset Z^*$.

If the division ring D is locally finite over its center, then consider the subgroup G_1 ,

generated by the elements $g_1 = 1, g_2, \dots, g_s$ and the subfield $D_1 = Z(g_2, \dots, g_s)$. The division ring D_1 is finite-dimensional over its center. The group D_1^* contains the normal divisor $G_2 = D_1^* \cap G \supset G_1$. From the preceding arguments it follows that G_2 is contained in the center of D_1^* . Consequently, G_1 is an abelian group. In view of the arbitrariness of G_1 , we obtain that G is an abelian normal divisor in D^* , which proves the theorem.

Theorem 2. *Let D be an arbitrary division ring, and let Z be its center. The factor group D^*/Z^* contains no nontrivial locally finite normal divisors.*

Let \bar{G} be a nontrivial locally finite normal divisor in the group D^*/Z^* : $Z^*, g_2Z^*, \dots, g_{s_1}Z^*$ is some collection of its elements. Then they generate a finite subgroup G_i , consisting of the cosets $Z^*, g_2Z^*, \dots, g_{s_1}Z^*$, where $g_1 = 1, g_2, \dots, g_s, \dots, g_{s_1}$ is a system of representatives of the cosets. Consider the subset of the division ring D consisting of elements

$$h = \sum_{i=1}^{s_1} z_i g_i,$$

where $z_i \in Z$, $i = 1, \dots, s_1$. These elements form a subfield which is finite-dimensional over Z . Therefore, if G is the full inverse image of the group \bar{G} in D^* under the natural homomorphism $D^* \rightarrow D^*/Z^*$, then the set of elements $a = \sum z_j g_j$, where $z_j \in Z$, $g_j \in G$, forms a subfield, locally finite-dimensional over its center. In view of the Cartan-Brauer-Hua theorem, this subfield coincides with D . Hence D is locally finite-dimensional over Z . From Theorem 1 it now follows that $G \subset Z^*$. This contradiction proves the theorem.

Remark. Theorem 2 can be obtained from Lemma 6 by applying a group-theoretic theorem proved in ⁽⁵⁾, p. 78.

Theorem 3. *Let D be a division ring of characteristic $p \neq 0$, and let G be a periodic normal divisor in the group D^* . Then $G \subset Z^*$.*

Proof. Denote by Ω the prime subfield of the division ring D . Consider the subring Y of the division ring D consisting of elements

$$y = \sum_{i=1}^n \alpha_i g_i, \tag{7}$$

where $\alpha_i \in \Omega$, $g_i \in G$. The minimal one among all possible n for which (7) holds will be called the length of the element y , if $y \neq 0$. We shall show that for every $y \neq 0$ there is a number $t = t(y)$, with $t > 0$, such that $y^{t(y)} = 1$.

Suppose it has been proved for all $y \neq 0$ whose lengths are less than n that $y^{t(y)} = 1$, where $t(y) > 0$. Assume that the element y has length $n > 1$. Then

$$y = \sum \alpha_i g_i = \alpha_1 g_1 \left(1 + \sum \alpha'_i g'_i \right) = \alpha_1 g_1 (1 + y_1),$$

where y_1 , by assumption, satisfies the condition $y_1^{t_1(y_1)} = 1$. Since the field Ω is finite, there exists a number $t_2 > 0$ such that $(1 + y_1)^{t_2} = 1$. Then, in the factor group D^*/G , the coset yG has finite order. It follows that the element y has finite order. Thus, all elements y of the ring Y satisfy the condition $y^{t(y)+1} = y$. From Jacobson's theorem (¹), p. 314) it follows that Y is commutative. Since $Y \supset G$, we have $G \subset Z^*$. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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