ON A GENERALIZATION OF THE CONCEPT OF PERIODIC EXTENSION

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Abstract

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MATHEMATICS

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ON A GENERALIZATION OF THE CONCEPT OF PERIODIC EXTENSION

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1. Let the complex numbers λ_k $(k = 1, 2, \ldots)$ be pairwise distinct. Associating with the sequence Λ, in which each λ_k occurs a (finite) number α_k of times, the sequence E_Λ of functions

$$e_{kj}(t) = (it)^{j-1} e^{i\lambda_k t} \qquad (j = 1, \ldots, \alpha_k;\ k = 1, 2, \ldots),$$

we denote by $L^2_\Lambda(-\sigma, \sigma)$ the closed vector subspace generated by the sequence E_Λ in $L^2(-\sigma, \sigma)$.

Definition 1. An **extension** of a function $f(t) \in L^2_\Lambda(-\sigma, \sigma)$ **with respect to the sequence** E_Λ will mean a function $F(t)$, defined on the whole real axis and coinciding with $f(t)$ almost everywhere on $(-\sigma, \sigma)$, whose restriction to each finite interval $(-\tau, \tau)$ belongs to the subspace $L^2_\Lambda(-\tau, \tau)$.

The periodic extension of a function $f(t) \in L^2(-\pi, \pi)$ is, according to this definition, the extension of the function $f(t)$ with respect to the sequence e^{ikt} $(k = 0, \pm 1, \pm 2, \ldots)$.

We shall study the following problem:

What conditions must the sequence Λ and the number $\sigma > 0$ satisfy in order that every function $f(t) \in L^2_\Lambda(-\sigma, \sigma)$ have a unique extension with respect to the sequence E_Λ?

The answer to this question is given by Theorems 1 and 2 of the present note. For what follows it is important that, by virtue of Banach' s theorem on open mappings, the posed problem may be given the following form:

What conditions must the sequence Λ and the number $\sigma > 0$ satisfy in order that for every $\tau > \sigma$ there exist a constant M, depending on τ, such that

$$\int_{-\tau}^{\tau} \left| \sum_{k,j} c_{kj} e_{kj}(t) \right|^2 dt \leq M \int_{-\sigma}^{\sigma} \left| \sum_{k,j} c_{kj} e_{kj}(t) \right|^2 dt \tag{1}$$

for every finite sequence of complex numbers c_{kj}.

Questions close in their formulation were studied earlier by A. F. Leont' ev (^{1}) and Zh. P. Kagan (^{2}).

2. Along with the functions $e_{kj}(t)$ we shall have to consider functions of a somewhat more general form; their construction is as follows.

Divide the sequence Λ into nonempty pairwise disjoint finite sets Λ_k ($k = 1, 2, ...$), denoting by ν_k the number of elements of the set Λ_k, and by $\lambda_{k,0}, \lambda_{k,1}, \dots, \lambda_{k,\nu_k-1}$ the elements themselves. The resulting partition will be denoted by $\hat{\Lambda}$.

Put

$$\hat{e}_{kj}(t) = \frac{1}{2\pi i} \int_{C_k} \frac{e^{izt}}{\omega_{kj}(z)} \, dz \qquad (j = 1, \dots, \nu_k;\ k = 1, 2, ...), \tag{2}$$

where

$$\omega_{kj}(\zeta) = \frac{1}{(j-1)!} (\zeta - \lambda_{k,0}) \dots (\zeta - \lambda_{k,j-1}), \tag{3}$$

and C_k is a contour enclosing all points of the set Λ_k.

It is verified directly that every linear combination of the functions $e_{kj}(t)$ is at the same time a linear combination of the $\hat{e}_{kj}(t)$, and conversely; thus, the sequence $\hat{E}_\Lambda$ of functions $\hat{e}_{kj}(t)$ $(j = 1, \dots, \nu_k;\ k = 1, 2, ...)$ and the sequence E_Λ generate one and the same vector subspace. For $\lambda_{k,0} = \lambda_k$, $\nu_k = \alpha_k$ $(k = 1, 2, ...)$, the sequence $\hat{E}_\Lambda$ coincides with E_Λ.

3. A family of points x_i $(i \in I)$ of a topological vector space E over the field of real or complex numbers will be called **completely free** if there exists in E a neighborhood of zero U such that for every $k \in I$ the neighborhood $x_k + U$ does not intersect the closed vector subspace generated by those x_i for which $i \neq k$. In order that a family of points x_i $(i \in I)$ of a locally convex space E be completely free, it is necessary and sufficient that there exist on E an equicontinuous family of linear forms f_i $(i \in I)$ forming with $(x_i)_{i \in I}$ a biorthogonal system.

By $\tau(\hat{E}_\Lambda)$ we shall denote the exact lower bound of those $\sigma > 0$ for which the sequence $\hat{E}_\Lambda$ is completely free in $L^2(-\sigma, \sigma)$. If the sequence $\hat{E}_\Lambda$ is not completely free in $L^2(-\sigma, \sigma)$ for any σ, we shall say that $\tau(\hat{E}_\Lambda) = \infty$.

We shall call a partition $\hat{\Lambda}$ **regular** if: 1) all numbers λ_k lie in some strip $|\operatorname{Im} \lambda| \leq h$; 2) the numbers ν_k are bounded in the aggregate; 3) for $\lambda \in \Lambda_k$, $\mu \in \Lambda_j$ $(k \neq j)$ the inequality $\inf |\lambda - \mu| > 0$ holds, where the lower bound is taken over all $0 < k, j < \infty$; 4) there exists a constant M such that $|\lambda_{k,0} - \lambda_{k,j}| \leq M$ $(j = 0, \dots, \nu_k - 1;\ k = 1, 2, ...)$.

Proposition 1. *If the partition* $\hat{\Lambda}$ *is regular, then* $\tau(\hat{E}_\Lambda) < \infty$.

Proposition 2. *For a given sequence* Λ, *the number* $\tau(\hat{E}_\Lambda)$ *has one and the same value* $\hat{\tau}_\Lambda$ *for any regular partition* $\hat{\Lambda}$.

We omit the proofs of these propositions.

4. **Definition 2.** The sequence Λ is called **regular** if: 1) all numbers λ_k $(k = 1, 2, ...)$ lie in some strip $|\operatorname{Im}\lambda| \le h$; 2) there exists a constant $N > 0$ such that no rectangle of the form

$$R_t = \{\lambda : \ t \le \operatorname{Re}\lambda \le t+1; \ |\operatorname{Im}\lambda| \le h\} \quad (-\infty < t < \infty) \tag{4}$$

contains more than N numbers λ_k, counted with their multiplicities α_k.

It is easy to show (see (3)) that the sequence Λ is regular if and only if it possesses at least one regular partition.

Theorem 1. *If the sequence* Λ *is regular, then for* $\sigma > \hat{\tau}_\Lambda$ *every function* $f(t) \in L^2_\Lambda(-\sigma, \sigma)$ *has a unique continuation with respect to the sequence* E_Λ.

The proof is based on the following propositions (cf. (4)).

Proposition 3. *If the partition* $\hat{\Lambda}$ *is regular, then for* $\sigma > 0$ *there exists a constant* B *such that*

$$\int_{-\sigma}^{\sigma} \left| \sum_{k,j} c_{kj} \hat{e}_{kj}(t) \right|^2 dt \le B \sum_{k,j} |c_{kj}|^2 \tag{5}$$

for any finite sequence of complex numbers c_{kj}.

Proposition 4. If the partition $\hat{\Lambda}$ is regular, then for $\sigma > \hat{\tau}_\Lambda$ there exists a constant A such that

$$A \sum_{k,j} |c_{kj}|^2 \le \int_{-\sigma}^{\sigma} \left| \sum_{k,j} c_{kj} \hat{e}_{kj}(t) \right|^2 dt, \tag{6}$$

for any finite sequence of complex numbers c_{kj}.

5. Theorem 2. If, for some $\sigma > 0$, every function $f(t) \in L^2_\Lambda(-\sigma, \sigma)$ has a unique extension with respect to the sequence E_Λ, then the sequence Λ is regular and $\sigma \ge \hat{\tau}_\Lambda$.

We shall briefly indicate the proof. First of all, for any $\tau > \sigma$ and some $M = M(\tau)$, inequality (1) must hold, whatever the finite sequence of complex numbers c_{kj}. In particular, taking $c_{nj} = \delta_{kn}\delta_{j0}$, we obtain:

$$|\operatorname{sh}(2\tau \operatorname{Im} \lambda_k)| \leq M |\operatorname{sh}(2\delta \operatorname{Im} \lambda_k)|$$

for all $k = 1, 2, \ldots$. Consequently, all the numbers λ_k lie in some strip $|\operatorname{Im} \lambda| \leq h$.

Suppose that condition 2 of Definition 2 is not satisfied. Then there will be sequences (t_k) and (ε_k) of positive numbers such that in each rectangle R_{t_k} there are at least k numbers $\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_k^{(k)}$ of the sequence Λ, for which $|\lambda_1^{(k)} - \lambda_i^{(k)}| \leq \varepsilon_k$ $(i = 1, 2, \ldots, k)$, and moreover $k! \varepsilon_k \to 0$ as $k \to \infty$. Put

$$\varphi_k(t) = \frac{(k-1)!}{2\pi i} \int_{L_k} \frac{e^{i\zeta t}\, d\zeta}{(\zeta - \lambda_1^{(k)}) \cdots (\zeta - \lambda_k^{(k)})},$$

where L_k is a contour enclosing the rectangle R_{t_k} and separated from its sides by a distance greater than one. Then

$$\varphi_k(t) = (it)^{k-1} e^{i\lambda_1^{(k)} t} + o(1)$$

as $k \to \infty$, which contradicts the inequality

$$\int_{-\tau}^{\tau} |\varphi_k(t)|^2 dt \leq M \int_{-\sigma}^{\sigma} |\varphi_k(t)|^2 dt,$$

valid by virtue of (1). Thus it has been proved that the sequence Λ is regular.

Let $\hat{\Lambda}$ be one of its regular partitions and let $\hat{E}_\Lambda$ be the sequence of functions $\hat{e}_{kj}(t)$ corresponding to this partition. Then for $\tau > \hat{\tau}_\Lambda$ it follows from inequality (1) that $\hat{E}_\Lambda$ is a completely free sequence in $L^2_\Lambda(-\sigma, \sigma)$, i.e. $\sigma \geq \hat{\tau}_\Lambda$.

Remark. Denote by T_Λ the exact lower bound of those $\sigma > 0$ for which every function $f(t) \in L^2_\Lambda(-\sigma, \sigma)$ has a unique extension with respect to the sequence E_Λ. The assertions of Theorems 1 and 2 may be combined by saying that $T_\Lambda = \hat{\tau}_\Lambda < \infty$ if the sequence Λ is regular, and $T_\Lambda = \infty$ if the sequence Λ is not regular.

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Note: Figure translations are in progress. See original paper for figures.

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