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Abstract

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ON THE ASYMPTOTICS OF INTEGER MATRICES OF ORDER n AND ON THE INTEGRAL INVARIANT OF THE GROUP OF UNIMODULAR MATRICES

(Presented by Academician I. M. Vinogradov, 12 VI 1963)

Let N be a sufficiently large natural number. Let

$$N = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

be the canonical decomposition of N. Consider an integer matrix of order n

$$X = (x_{ij}), \qquad \det X = N,$$

and associate with it the matrix

$$\widetilde{X} = \left(\frac{x_{ij}}{N^{1/n}}\right) = (\tilde{x}_{ij}), \qquad \det \widetilde{X} = 1.$$

We shall denote by the symbol Ω some Jordan-measurable domain among the unimodular matrices of order n, and by the symbol $\operatorname{mes}\Omega$ the Haar measure of this domain. Further, denote by D_k the greatest common divisor of the minors of order k of the matrix X.

We shall consider the surface $\det X = N$. On this surface we shall call a point **primitive** if $D_1 = 1$, and **completely primitive** if $D_{n-1} = 1$.

Theorem 1. The number of integer points $f(\Omega, N, n)$ on the surface $\det X = N$ with $\widetilde{X} \in \Omega$ satisfies the asymptotic formula

$$f(\Omega, N, n) \sim \frac{\operatorname{mes}\Omega}{\zeta(2)\cdots\zeta(n)} \prod_{p_i^{k_i}} \frac{(p_i^{k_i+1}-1)(p_i^{k_i+2}-1)\cdots(p_i^{k_i+n-1}-1)}{(p_i-1)(p_i^2-1)\cdots(p_i^{n-1}-1)}$$

as $N \to \infty$, with Ω fixed.

This theorem is an immediate consequence of the following theorem.

Theorem 2. The number of completely primitive points $f_{n-1}(\Omega, N, n)$ on the surface $\det X = N$ with $X \in \Omega$ satisfies the asymptotic formula

$$f(\Omega, N, n) \sim \frac{\operatorname{mes}\Omega}{\zeta(2)\cdots\zeta(n)} \frac{N^n}{\varphi(N)} \sum_{d|N} \frac{\mu(d)}{d^n}$$

as $N \to \infty$, with Ω fixed.

Theorem 2 is proved with the aid of a lemma, which is also of independent interest.

Lemma. If a completely primitive matrix of order $(n-1)$

$$B = (b_{ij}), \qquad \det B = q,$$

is such that

$$|b_{ij}| \leq q^{\frac{1}{n-1}+\varepsilon_n}, \qquad \text{o.n.d. } (q, N) \leq q^{\eta_0},$$

then the number of distinct solutions of the equation

$$\begin{vmatrix} B & \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{n-1} \end{pmatrix} \\ (u_n \ \cdots \ u_{2(n-1)}) & t \end{vmatrix} = N$$

in the variables $u_1,\ u_2, \dots, u_{n-1},\ u_n, \dots, u_{2(n-1)},\ t$ for fixed B, with the condition

$$\gcd(M_{n1}, M_{n2}, \dots, M_{n\,n-1}, q) = 1,$$

where M_{ij} is the minor of the element a_{ij} ($a_{ij} = b_{ij}$ for $i, j < n$; $a_{ij} = u_k$ otherwise), and, for

$$a_j q^{\frac{1}{n-1}} < u_j \leq a'_j q^{\frac{1}{n-1}}, \qquad a'_j - a_j \geq q^{-\varepsilon_1}, \qquad (j = 1, 2, \dots, 2(n-1)),$$

there is

$$R = \varphi(q) \prod_{j=1}^{2(n-1)} (a'_j - a_j) \left(1 + O\left(q^{-\left(\frac{1}{(n-1)(4n-1)} - \frac{\varepsilon_1+\eta_0}{4n-1} - \frac{4(n-1)}{4n-1}(\varepsilon_0+\varepsilon_1) - \zeta\right)}\right)\right),$$

where ζ is an arbitrarily small positive quantity; the constant in the remainder depends only on ζ.

In the course of proving this lemma, an interesting arithmetical observation emerges.

If $\gcd(q, N) = 1$, then $\gcd(q, M_{nj})$ and $\gcd(q, M_{in})$ $(i = 1, 2, \dots, n-1)$ are invariants of the solutions of the indeterminate equation

$$\begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1\,n-1} & u_1 \\ b_{21} & b_{22} & \cdots & b_{2\,n-1} & u_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{n-1\,1} & b_{n-1\,2} & \cdots & b_{n-1\,n-1} & u_{n-1} \\ u_n & u_{n+1} & \cdots & u_{2(n-1)} & t \end{vmatrix} = N$$

in the variables $u_1, u_2, \dots, u_{n-1}, u_n, \dots, u_{2(n-1)}, t$.

Our lemma is proved with the aid of the well-known lemma of I. M. Vinogradov on Fourier series[(1)] and estimates of A. Weil[(2)].

It follows from Theorem 1 that the volume V_n of the quotient space of the group of unimodular matrices by the subgroup of integral unimodular matrices of order n is

$$V_n = \zeta(2)\zeta(3)\cdots\zeta(n).$$

For the proof it suffices to note that: 1) in the space of the unimodular group one can always choose a fundamental domain so that it is quadrable in the sense of Jordan; 2) the expression

$$\prod_{p_i^{k_i}} \frac{(p_i^{k_i+1}-1)(p_i^{k_i+2}-1)\cdots(p_i^{k_i+n-1}-1)}{(p_i-1)(p_i^2-1)\cdots(p_i^{n-1}-1)}$$

is nothing other than the number of left (right) non-associated integral matrices of order n with determinant N.

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References

[1] I. M. Vinogradov, *Selected Works*, 1952.
[2] A. Weil, Proc. Nat. Akad. USA, **34**, 204 (1948).

Note: Figure translations are in progress. See original paper for figures.