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Abstract

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ON ULTRAPARABOLIC EQUATIONS

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In this paper we consider equations of the form

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = a_0 \frac{\partial u}{\partial t} + \sum_{k=1}^{m} a_k \frac{\partial u}{\partial y_k} + bu + f, \qquad (1)$$

where

$$\sum_{i,j=1}^{n} a_{ij}(x,y,t)\xi_i\xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2, \qquad \mu = \text{const} > 0, \qquad a_0(x,y,t) > \lambda > 0.$$

We shall call such equations **ultraparabolic**. Interest in equations of the form (1) arose in connection with the works of A. N. Kolmogorov ([1,2]), where certain problems of probability theory lead to the consideration of such equations. Equations of this type also occur in boundary-layer theory, in the theory of Brownian motion, etc. (see ([3,4])). In the case $n = 1$, equation (1) was studied in ([5,6]). The solution of the boundary-value problem in these works was obtained by the Rothe method.

In the present paper we shall obtain a generalized solution of the boundary-value problem and of the Cauchy problem for equation (1), and prove uniqueness of the solution of these problems.

1. Let E_n be the space of points with coordinates $(x_1, \dots, x_n)$, and E_m the space $(y_1, \dots, y_m)$. By (x, y, t) we shall denote points of the space $E_n \times E_m \times E_1$. Let Ω be a domain in E_n with twice continuously differentiable boundary σ, and let R be the parallelepiped $\{0 < y_k < \beta_k,\ 0 < t < T\}$, $k = 1, \dots, m$; $G = \Omega \times R$ is a domain in $E_n \times E_m \times E_1$. By $W_{2,x}^1(G)$ we denote the Hilbert space of functions with norm

$$\|u\|_{W_{2,x}^1}^2 = \int_G \left[u^2 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right] dG, \qquad \|u\|_{L_2(G)}^2 = \int_G u^2 \, dG.$$

We define the norm of the space $W_{2,y,t}^1(G)$ by the equality

$$\|u\|^2_{W^1_{2,y,t}} = \int_G \left[u^2 + \sum_{i=1}^{m} \left(\frac{\partial u}{\partial y_i} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dG;$$

the norm of the space $H(G)$ by the equality

$$\|u\|^2_{H(G)} = \int_G \left[u^2 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 + \sum_{i=1}^{m} \left(\frac{\partial u}{\partial y_i} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 + \sum_{i,j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \right.$$

$$\left. + \sum_{i,k=1}^{n,m} \left(\frac{\partial^2 u}{\partial y_k \partial x_i} \right)^2 + \sum \left(\frac{\partial^2 u}{\partial x_i \partial t} \right)^2 \right] dG,$$

and the norm of the space $W^1_2(G)$ by the equality

$$\|u\|_{W^1_2(G)} = \|u\|_{W^1_{2,x}} + \|u\|_{W^1_{2,y,t}}.$$

2. We shall assume:

I. The a_{ij} are continuous in the closed domain $\overline{G}$ and have bounded derivatives of the form $\partial a_{ij}/\partial x_i$, $\partial a_{ij}/\partial y_k$, $\partial a_{ij}/\partial t$, $\partial^2 a_{ij}/\partial x_i \partial x_j$, $\partial^2 a_{ij}/\partial x_i \partial y_k$, $\partial^2 a_{ij}/\partial x_i \partial t$, $i, j = 1, \dots, n;\ k = 1, \dots, m$.

II. The b_i are bounded in $\overline{G}$, the coefficient b is bounded below, $b \subset L_2(G)$, $a_k \subset L_2(G)$, and the derivatives $\partial a_s/\partial y_k$, $\partial a_s/\partial t$, $\partial b/\partial y_k$, $\partial b/\partial t$, $\partial b_i/\partial y_k$, $\partial b_i/\partial t$, $\partial a_0/\partial y_k$, $\partial a_0/\partial t$ $(i = 1, \dots, n';\ s, k = 1, 2, \dots, m)$ are bounded in $\overline{G}$.

III. The a_k do not change sign for $y_k = 0$ and $y_k = \beta_k,\ k = 1, \dots, m$.

Divide the boundary S of the domain G into two parts: $S = S_1 + S_2$. If $a_k \geqslant 0$ for $y_k = 0$ $(k = 1, \dots, m)$, then assign the boundary $y_k = 0$ to S_1, otherwise to S_2; if $a_k \leqslant 0$ for $y_k = \beta_k$, then assign the boundary $y_k = \beta_k$ to S_1, otherwise to S_2. Assign the boundary $\sigma \times R$ and $t = 0$ to S_1, and the boundary $t = T$ to S_2.

Theorem 1. *Suppose that the coefficients of equation (1) satisfy conditions I, II, III. If the function $f(x, y, t) \subset W^1_{2,y,t}(G)$ and $f|_{S_1 - \sigma \times R} = 0$, then there exists a unique function of the class $H(G)$ which almost everywhere in G satisfies equation (1) and assumes zero values on S_1.*

We indicate the main ideas of the proof of this theorem. Consider the domain $G_1 = \Omega \times R_1$, where R_1 in $E_m \times E_1$ is given by the inequalities $\{0 \leqslant t \leqslant T + \delta,\ \delta > 0,\ \alpha_k \leqslant y_k \leqslant \gamma_k,\ k = 1, \dots, m\}$ and $\alpha_k \leqslant 0,\ \gamma_k \geqslant \beta_k$. If the boundary $y_k = 0$ belongs to S_1, then $\alpha_k = 0$, otherwise $\alpha_k = -\delta$. Similarly, if the boundary $y_k = \beta_k \subset S_1$, then $\gamma_k = \beta_k$, otherwise $\gamma_k = \beta_k + \delta$.

Consider in G_1 the auxiliary equation of elliptic type:

$$\sum_{i,j=1}^{n} a_{ij}^{\varepsilon}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{k=1}^{m}\varepsilon\frac{\partial^2 u}{\partial y_k^2}+\varepsilon\frac{\partial^2 u}{\partial t^2}+\sum_{i=1}^{n} b_i^{\varepsilon}\frac{\partial u}{\partial x_i} = a_0^{\varepsilon}\frac{\partial u}{\partial t}+\sum_{k=1}^{m} a_k^{\varepsilon}\frac{\partial u}{\partial y_k}+a_0^{\varepsilon}\frac{\partial u}{\partial t}+b^{\varepsilon}u+f^{\varepsilon}, \tag{2}$$

where $a_{ij}^{\varepsilon}, b_i^{\varepsilon}, a_k^{\varepsilon}, a_0^{\varepsilon}, f^{\varepsilon}, b^{\varepsilon}$ are infinitely differentiable functions converging in the mean as $\varepsilon \to 0$ in G to the corresponding coefficients of equation (1), $\varepsilon > 0$. At the same time, all derivatives of the coefficients of equation (2) indicated in conditions I and II are uniformly bounded with respect to ε. It is easy to see that the coefficients of equation (2) can be chosen so that the following conditions are satisfied:

1)
$$\sum_{i,j=1}^{n} a_{ij}^{\varepsilon}\xi_i\xi_j \geqslant \frac{\mu}{2}\sum_{i=1}^{n}\xi_i^2 \quad \text{for all } (x,y,t)\subset G_1.$$

2) $a_0^{\varepsilon}(x,y,T+\delta)=0$, $a_0^{\varepsilon}>0$ in G, $a_k^{\varepsilon}|_{y_k=\alpha_k}=0$, if $\alpha_k<0$, and $a_k^{\varepsilon}|_{y_k=\gamma_k}=0$, if $\gamma_k>\beta_k$. On the boundary of G_1 determined by the equalities $y_k=\alpha_k=0$ or $y_k=\gamma_k=\beta_k$, the functions a_k^{ε} and a_k have the same sign.

3) The function f^{ε} is finite in G_1 and the norms
$$\|f^{\varepsilon}\|_{W^1_{2,y,t}(G_1)}$$
are uniformly bounded with respect to ε.

4) $b^{\varepsilon}>b_0>0$, if $b>b_0$.

For equation (2) consider in the domain G_1 the following boundary-value problem:

$$u_\varepsilon = 0 \quad \text{on the boundary of } G_1. \tag{3}$$

It can be shown that the solution u_ε of problem (2), (3) has all second-order derivatives continuous in $\overline{G}_1$. For the solution u_ε there are a priori estimates

$$\int_{G_1}\left[\sum_{i=1}^{n}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)^2+\varepsilon\sum_{k=1}^{m}\left(\frac{\partial u_\varepsilon}{\partial y_k}\right)^2+\varepsilon\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2+u_\varepsilon^2\right]dG_1 \leqslant M\|f^{\varepsilon}\|^2_{L_2(G_1)}; \tag{4}$$

$$\int_{G_1}\left[\sum_{i,j=1}^{n}\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\right)^2\right]dG_1+\int_{G_1}\left[\sum_{\substack{i=1,\dots,n\\k=1,\dots,m}}\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial y_k}\right)^2+\sum_{i=1}^{n}\left(\frac{\partial^2 u_\varepsilon}{\partial x_i\partial t}\right)^2+\varepsilon\sum_{k,\nu=1}^{m}\left(\frac{\partial^2 u_\varepsilon}{\partial y_\nu\partial y_k}\right)^2+\right.$$
$$\left.+\sum_{k=1}^{m}\varepsilon\left(\frac{\partial^2 u_\varepsilon}{\partial y_k\partial t}\right)^2+\varepsilon\left(\frac{\partial^2 u_\varepsilon}{\partial t^2}\right)^2+\sum_{k=1}^{m}\left(\frac{\partial u_\varepsilon}{\partial y_k}\right)^2+\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2\right]dG_1 \le M\|f^{\varepsilon}\|^2_{W^1_{2,y,t}(G_1)}, \tag{5}$$

where M does not depend on ε. Estimate (4) can be obtained by multiplying (2) by u_ε, integrating this equality over G_1, and transforming it by integration by parts. The second integral on the left-hand side of (5) can be estimated if equation (2) is multiplied by $\partial^2 u_\varepsilon/\partial y_k^2$ $(k = 1, \dots, m)$ and $\partial^2 u_\varepsilon/\partial t^2$, the resulting $m+1$ equalities are integrated over G_1, added, and some of the resulting integrals are transformed by integration by parts. The first integral on the left-hand side of (5) is then estimated in the same way as the second derivatives of an elliptic-type equation are estimated in the L_2 norm (see, for example, (7), § 2). With the aid of estimates (4) and (5) it is easy to show that there exists a limit of u_ε as $\varepsilon \to 0$ along a certain subsequence ε_k, and the limiting function $u \subset H(G)$ almost everywhere satisfies equation (1) in G and the condition $u|_{S_1} = 0$. The uniqueness of such a solution is proved with the aid of an energy inequality.

3. A function $u(x, y, t) \subset W^1_{2,x}(G)$ is called a **weak generalized solution** of equation (1) in the domain G with the condition $u|_{S_1} = 0$ if, for every function $\Phi \subset W^1_2(G)$ satisfying the conditions $\Phi|_{\sigma\times R} = 0$, $\Phi|_{S_2} = 0$, the integral identity holds:

$$\int_G \Bigg\{ -\left[\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j} + \frac{\partial a_{ij}}{\partial x_j}\frac{\partial u}{\partial x_i}\Phi\right] + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i}\Phi +$$

$$+ \sum_{k=1}^m \left(a_k \frac{\partial \Phi}{\partial y_k} u + \frac{\partial a_k}{\partial y_k}\Phi u\right) + a_0 \frac{\partial \Phi}{\partial t} u + \frac{\partial a_0}{\partial t} u\Phi - (bu + f)\Phi \Bigg\} dG = 0.$$

Theorem 2. Suppose the coefficients of equation (1) satisfy conditions I, II, III. If $f \subset L_2(G)$, then there exists a unique weak generalized solution of equation (1) with the condition $u|_{S_1} = 0$.

The existence of such a solution is proved using the solutions u_ε of problem (2), (3) and the a priori estimate (4). The uniqueness of such a solution is easily obtained by using the existence of a solution of the equation adjoint to (1), which can be constructed according to Theorem 1.

4. Consider the Cauchy problem for equation (1) in the layer $Q\{0 < t \leq T,\ (x, y) \subset E_n \times E_m\}$ with the initial condition

$$u(x, y, 0) = 0, \qquad (x, y) \subset E_n \times E_m. \tag{6}$$

A function $u(x, y, t) \subset W^1_2(Q)$ will be called a **generalized solution** of problem (1), (6) if $u|_{t=0} = 0$ and, for every function Φ from $W^1_2(Q)$ that is finite in x and y, the integral identity holds

$$\int_Q \Bigg\{ -\left[\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j} + \frac{\partial a_{ij}}{\partial x_j}\frac{\partial u}{\partial x_i}\Phi\right] + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i}\Phi -$$

$$- \left[\sum_{k=1}^m a_k \frac{\partial u}{\partial y_k} + a_0\frac{\partial u}{\partial t} + bu + f\right]\Phi \Bigg\} dQ = 0.$$

Theorem 3. If the coefficients b_i, b, and all derivatives entering conditions I, II are bounded in Q, and $f \subset W^1_{2,y,t}(Q)$, $f|_{t=0} = 0$, then there exists a unique generalized solution of the Cauchy problem (1), (6).

To prove this theorem one considers the solution u_ε of equation (2) in the domain $G_\varepsilon = \Omega_\varepsilon \times R_\varepsilon$, equal to zero on the boundary G_ε, where

$$\Omega_\varepsilon = \left\{ \sum_{i=1}^{n} x_i^2 \le A_\varepsilon \right\}, \qquad R_\varepsilon = \{|y_k| \le \beta^\varepsilon,\ 0 \le t \le T + \delta\}.$$

The coefficients (2) approximate the coefficients of equation (1) as $\varepsilon \to 0$. Denote the second integral in the left-hand side of (5) by I_1. For solutions u_ε in the domain G_ε, estimate (4) is valid, as is the estimate

$$I_1 \le M \|f^\varepsilon\|^2_{W^1_{2,y,t}(G_\varepsilon)},$$

where M does not depend on ε. Letting ε tend to zero, and A_ε and β^ε to infinity, and choosing a sequence u_ε that converges weakly in W^1_2 in any finite domain, we obtain in the limit a solution of problem (1), (6). The uniqueness of such a solution follows from the energy inequality.

From the theorem proved there follows, in particular, solvability under the indicated conditions of the Cauchy problem for the equation of Brownian motion in phase space:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m} x_i \frac{\partial u}{\partial y_i} + a \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{m} b_i \frac{\partial u}{\partial x_i} + f(x, y, t), \qquad a > 0.$$

One may consider the Cauchy problem for equation (1) in the class of functions increasing in x and y at infinity, analogously to how this is done in § 10 of paper (7).

5. Another formulation of the Cauchy problem is also possible, when the variables $(t, y_1, \dots, y_\nu)$, $\nu \le m$, play the same role as the time t, while $(x_1, \dots, x_n, y_{\nu+1}, \dots, y_m)$ are regarded as spatial variables. In this case the solution is sought in the domain $\{0 \le t \le T,\ 0 \le y_s \le \beta_s,\ s = 1, \dots, \nu\}$, and the initial conditions must be prescribed at $t = 0$ and on those parts of the boundary $y_s = 0$ and $y_s = \beta_s$ that belong to S_1. For such a problem a theorem analogous to Theorem 3 is valid.

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[1] A. N. Kolmogoroff, *Ann. of Math.*, **35**, No. 1, 116 (1934).
[2] A. N. Kolmogoroff, *Math. Ann.*, **108**, 150 (1937).
[3] S. Chandrasekhar, *Stochastic Problems in Physics and Astronomy*, IL, 1947.
[4] K. Ito, *Collection of Translations. Mathematics*, **3**, No. 5, 131 (1959).
[5] N. S. Piskunov, *Matem. sbornik*, **7**, No. 3, 385 (1940).
[6] N. S. Piskunov, *Matem. sbornik*, **1**(43), 931 (1936).
[7] A. M. Il' in, A. S. Kalashnikov, O. A. Oleinik, *Uspekhi Mat. Nauk*, **17**, issue 3 (105), 3 (1962).

Note: Figure translations are in progress. See original paper for figures.