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Abstract

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MATHEMATICS

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ON LINEAR DIOPHANTINE APPROXIMATIONS

(Presented by Academician I. M. Vinogradov on 26 VII 1962)

From Minkowski' s lemma on linear forms (see [1], p. 36) the following assertion obviously follows:

Lemma. *Whatever three real numbers $\omega_1, \omega_2, \omega_3$ we take, there exists an infinite number of solutions in integers of the system of inequalities*

$$|\omega_1 x - y| \leq \frac{1}{\sqrt{x}}, \qquad |\omega_2 x + \omega_3 y - z| \leq \frac{1}{\sqrt{x}}.$$

With the help of the device set forth in the book [2], p. 24, the assertion of the lemma can be strengthened.

Lemma. *Whatever three real numbers $\omega_1, \omega_2, \omega_3$ we take, there exists an infinite number of solutions in integers of the system of inequalities*

$$|\omega_1 x - y| \leq \frac{2}{3\sqrt{x}}, \qquad |\omega_2 x + \omega_3 y - z| \leq \frac{2}{3\sqrt{x}}.$$

We shall call a system of three real numbers $\omega_1, \omega_2, \omega_3$ K-supernormal if, for every $\varepsilon > 0$, the system of inequalities

$$|x\omega_1 - y| < \frac{\varepsilon}{\sqrt{x}}, \qquad |x\omega_2 + y\omega_3 - z| < \frac{\varepsilon}{\sqrt{x}}$$

has an infinite number of solutions in integers. We shall call a system of numbers $\omega_1, \omega_2, \omega_3$ K-normal if it is not K-supernormal.

Theorem 1. *If for a system of three numbers $\omega_1, \omega_2, \omega_3$ one can find three integers $x_1^{(0)} \neq 0$, $x_2^{(0)}$, $x_3^{(0)}$ such that*

$$x_1^{(0)}(\omega_1\omega_3 + \omega_2) + x_2^{(0)}\omega_1 + x_3^{(0)} = 0,$$

then the system of numbers $\omega_1, \omega_2, \omega_3$ *is* K*-supernormal.*

Theorem 2. *Let* θ *be a real irrationality of degree three,* $\omega_1 = \theta$*,* ω_3 *any real number,* $\omega_2 = \theta^2 - \omega_3\theta$*. The system of numbers* $\omega_1, \omega_2, \omega_3$ *is a* K*-normal system of numbers.*

In this circle of ideas there also lies the following metric theorem.

Theorem 3. *Consider the system of Diophantine inequalities*

$$|\omega_1 x - y| \leq \frac{c}{\sqrt{x}\ln^\gamma x}, \qquad |\omega_2 x + \omega_3 y - z| \leq \frac{c}{\sqrt{x}\ln^\gamma x}. \tag{1}$$

If $\gamma > 1/2$ *(strictly), then the measure of the set of points* $(\omega_1, \omega_2, \omega_3)$ *of the unit cube for which inequality (1) has an infinite number of solutions in integers is equal to zero. If* $0 < \gamma \leq 1/2$*, then the measure of the set of points* $(\omega_1, \omega_2, \omega_3)$ *of the unit cube for which inequality (1) has an infinite number of solutions in integers is equal to one.*

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CITED LITERATURE

1. B. A. Venkov, *Elementary Number Theory*, 1937.

2. J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Moscow, 1961.

Note: Figure translations are in progress. See original paper for figures.