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# Yu. I. Manin

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**Abstract**

**Full Text**

**Yu. I. Manin**

## **On the Arithmetic of Rational Surfaces**

*(Presented by Academician I. M. Vinogradov on 14 III 1963)*

### **1.**

By a rational surface we shall mean any nonsingular projective algebraic surface birationally equivalent to the projective plane over the algebraic closure of the ground field. The arithmetic properties of curves of genus zero are described by classical results: if over every completion of the field of constants a curve contains a rational point, then it contains a rational point over the ground field; two curves that are isomorphic over every completion of the field of constants are isomorphic; according as a rational point exists or does not exist, a curve is isomorphic either to the projective line or to a curve of the second degree. (We shall refer to the first two assertions respectively as the first and second Minkowski-Hasse principles.) There is no analogous theory of rational surfaces. The birational aspect of the problem is complicated by the fact that the extensive group of birational automorphisms of the plane has been little studied; moreover, in the arithmetic of general rational surfaces (in contrast to Severi-Brauer surfaces) the passage “from local to global” is impossible: Swinnerton-Dyer <sup>(1)</sup> constructed a cubic surface not satisfying the first Minkowski-Hasse principle, and below we propose examples of violation of the second principle. It is of interest to determine how broad a class of surfaces does not yet exhibit these pathologies. In § 2 we prove the following generalization of Châtelet’s theorem on Severi-Brauer surfaces: *both Minkowski-Hasse principles are valid for relatively minimal models of rational surfaces.* § 4 is devoted to the computation of the local and global zeta-functions of a rational surface.

### **2.**

We shall freely use the results of Nagata’s paper <sup>(3)</sup>. The class of rational surfaces contains a countable set of distinct minimal models, consisting of the projective plane  $P^2$ , the quadric  $P^1 \times P^1$ , and the series of ruled surfaces  $F_n$ ,  $n \geq 2$ . Statements of arithmetic character concern forms of these models over nonclosed fields—the fields of constants (see <sup>(4)</sup>), and we constantly use the fundamental correspondence between these forms and elements of a certain set of Galois cohomologies. We shall restrict ourselves to the surfaces  $F_n$ , since the forms of  $P^2$  are Severi-Brauer surfaces, and the forms of  $P^1 \times P^1$  are nonsingular quadrics: for both, the validity of the Minkowski-Hasse principles is known. Let  $F$  be some form of the surface  $F_n$  over a perfect field  $k$ . On  $F$  lies a nonsingular

curve of genus zero  $b \subset F$ , whose self-intersection index is  $(b^2) = -n$ , and this is the only curve with such an index. Consequently, for any automorphism  $\sigma$  of the algebraic closure of the field  $k$  over  $k$  we have  $b^\sigma = b$ , so that the curve  $b$  is defined over  $k$ . Moreover, there exists, and is uniquely determined, a regular mapping  $\pi : F \rightarrow b$ , identical on  $b$ , and such that the inverse image of any point of the base  $b$  is a nonsingular irreducible curve of genus zero. By uniqueness,  $\pi$  is also defined over the field  $k$ . Hence follows the following result, reducing the verification of the first principle to a known case:

**Theorem 1.** *A form  $F$  of the surface  $F_n$  over a perfect field of constants has a rational point if and only if the base  $b \subset F$  has a rational point.*

(If a point  $Q \in F$  is rational, then the point  $\pi(Q) \in b$  is also rational.)

The verification of the second principle rests on a similar result:

**Theorem 2.** *Forms of the surface  $F_n$  over a perfect field of constants  $k$  are in one-to-one correspondence with forms of the projective line over  $k$ : to each form of the surface there corresponds the form of its base.*

**Proof.** Let  $G$  be the algebraic group of automorphisms of the surface  $F_n$ . Every automorphism takes the base  $b$  to itself, which determines a homomorphism

$$j : G \rightarrow PGL(2)$$

from the group  $G$  to the group of automorphisms of the base. There exists a homomorphism—a section

$$s : PGL(2) \rightarrow G,$$

so that the group  $G$  is the semidirect product of the normal divisor

$$\text{Ker } j = \Gamma$$

and the subgroup  $s(PGL(2))$ , and  $j$  is an epimorphism. The group  $\Gamma$ —the stationary subgroup of the base  $b$ —contains, as a normal divisor, the direct sum of  $n + 1$  additive groups  $(G_a)^{n+1}$ , the factor group by which is isomorphic to the multiplicative group  $G_m$ . The exact sequence

$$1 \rightarrow (G_a)^{n+1} \rightarrow \Gamma \rightarrow G_m \rightarrow 1$$

shows that

$$H^1(k, \Gamma) = 1.$$

From the exact sequence

$$1 \rightarrow \Gamma \rightarrow G \xrightarrow{j} PGL(2) \rightarrow 1$$

and the existence of the section  $s$ , it follows that

$$j_* : H^1(k, G) \rightarrow H^1(k, PGL(2))$$

is an isomorphism. This is precisely the assertion of the theorem.

We have made essential use of information about the structure of the group  $G$ . Let us briefly describe how to compute this group <sup>(5)</sup>. Consider the projective line  $P^1$  with a system of homogeneous coordinates  $(u_0, u_1)$  and the affine plane  $A^2$  with a system of coordinates  $(t_1, t_2)$ ; let  $A = A^2 - (0, 0)$ . Define an action of the multiplicative group on

$$V = P^1 \times A$$

by setting

$$\lambda\{(u_0, u_1), (t_1, t_2)\} = \{(u_0, \lambda^n u_1), (\lambda t_1, \lambda t_2)\},$$

where  $n$  is any integer. The space  $V/G_m$  is isomorphic to  $F_n$  for  $n \geq 2$ : the map

$$\{(u_0, u_1), (t_1, t_2)\} \rightarrow \{(u_1, u_0 t_2^n, u_0 t_1^n), (t_1, t_2)\}$$

establishes the identity of  $V/G_m$  with  $F_n$  in Andreotti's definition <sup>(6)</sup>. The base  $b \subset F_n$  consists of the orbits of the subset

$$(1, 0) \times A \subset V,$$

and the projection  $F_n \rightarrow b$  is induced by the projection

$$P^1 \times A \rightarrow A.$$

The group  $G$  is induced by the group of automorphisms of  $V$  compatible with the action of  $G_m$ :

$$\{(u_0, u_1), (t_1, t_2)\} \rightarrow \{(u_0, \mu u_1 + u_0 \Phi_n(t_1, t_2)), (t_1, t_2)L\}.$$

Here  $\mu$  is a nonzero constant,  $\Phi_n$  is a form of degree  $n$ , and  $L$  is a nonsingular matrix of order two. The subgroup

$$s(PGL(2)) \subset G$$

is the image of the group of those automorphisms of  $V$  for which

$$\mu = 1, \quad \Phi_n = 0;$$

$\Gamma$  corresponds to automorphisms with the identity matrix  $L$ ; finally, the subgroup

$$(G_a)^{n+1} \subset \Gamma$$

is induced by automorphisms with  $\mu = 1$ .

3. We shall now show that, in the class of all rational surfaces, the second Minkowski-Hasse principle is false. We shall need the following.

**Lemma 1.** Let  $P^2$  be the projective plane over an algebraically closed field  $k$ ; let

$$Q_1, \dots, Q_n$$

be points lying on a single line

$$P^1 \subset P^2, \quad n \geq 2;$$

and let  $F$  be the surface obtained from  $P^2$  by quadratic transformations with centers at the points

$$Q_1, \dots, Q_n.$$

Then every biregular automorphism of the surface  $F$  is induced by a projective transformation of the plane  $P^2$  permuting the points

$$Q_1, \dots, Q_n.$$

**Proof.** Let  $b, c_i \subset F$  be the proper transforms of the line  $P^1$  and of the points  $Q_i$ , respectively. We have

$$(b^2) = 1 - n, \quad (c_i^2) = -1.$$

There are no other irreducible curves  $c$  with negative self-intersection on the surface  $F$ . Indeed, otherwise such a curve would be the image of some irreducible curve of degree  $m$ , passing through the points  $Q_i$  with multiplicities  $m_i$ . Since all the points  $Q_i$  lie on one line, we have

$$m \geq \sum_{i=1}^n m_i,$$

whence

$$(c^2) = m^2 - \sum_{i=1}^n m_i^2 \geq 0.$$

It follows that every biregular automorphism of the surface  $F$  permutes the curves  $c_i$ , and takes the curve  $b$  to itself. Thus such an automorphism induces a biregular automorphism of the open set

$$P^2 - \bigcup_{i=1}^n Q_i;$$

the corresponding birational automorphism of the plane is regular at the points  $Q_i$  and permutes them. The lemma is proved.

For the construction of examples we shall choose the points  $Q_i$  in a special way. Introduce on the line  $P^1$  an affine coordinate  $x$ . Suppose that in this coordinate system the point  $Q_i$  is  $\zeta^i$ , where  $\zeta$  is a primitive root of uni-

...of degree  $n$ . Since the cycle  $\sum_{i=1}^n Q_i \in P^2$  is rational over the field  $Q$  in some system of coordinates, there exists a form of the surface  $F$  defined over  $Q$ .

Let  $G$  be the automorphism group of the surface  $F$ , and  $G_0$  the automorphism group of the line  $P^1$  permuting the points  $Q_i$ . The restriction homomorphism  $G \rightarrow G_0$  is an epimorphism, for there exists a section  $G_0 \rightarrow G$ . Its kernel—the stationary subgroup  $P^1$ —is isomorphic to the group of affine transformations of the form  $(x, y) \rightarrow (\alpha x + \beta, \alpha x + \gamma)$ . The Galois cohomology of this kernel is trivial; therefore  $H^1(k, G) \approx H^1(k, G_0)$ . The group  $G_0$  contains the normal divisor  $\Gamma_n : x \rightarrow \zeta^i x, i = 0, \dots, n - 1$ . The whole group  $G_0$  is generated by  $\Gamma_n$  and the involution  $x \rightarrow x^{-1}$ . For  $n \geq 3$  the corresponding segment of the exact sequence

$$H^0(k, G_0) \rightarrow H^0(k, Z_2) \rightarrow H^1(k, \Gamma_n) \rightarrow H^1(k, G_0)$$

shows that  $H^1(k, \Gamma_n)$  embeds in the set  $H^1(k, G_0) = H^1(k, G)$ , for the first homomorphism in the line is an epimorphism. From the “exact Kummer sequence” it follows that  $H^1(k, \Gamma_n) \approx k^*/(k^*)^n$ . Now we can use, for the construction of examples, the exceptional cases in Grunwald’s theorem (7). In these cases the field  $k$  contains an element  $a$  which is locally everywhere an  $n$ -th power, but is not an  $n$ -th power as a whole. The cohomology class  $a^* \in H^1(k, G)$  corresponding to this element determines a form of the surface  $F$ , isomorphic to  $F$  over all completions of the field  $k$ , but not isomorphic over  $k$ . The simplest example is obtained by taking  $k = Q(\sqrt{7})$ ,  $n = 8$ .

4. The following result was predicted by A. Weil (8).

**Theorem 3.** *Let  $F$  be a nonsingular rational surface defined over a finite field of  $q$  elements, and let  $N_a$  be the number of points on  $F$  rational over the field of  $q^a$  elements. Then*

$$N_a = q^{2a} + q^a \operatorname{Tr} \Phi^a + 1,$$

where  $\Phi$  is the endomorphism of the group  $C(F)$  of divisor classes of the surface  $F$  (over the closure of the ground field), up to numerical equivalence, induced by the Frobenius endomorphism of the surface  $F$ .

**Corollary.**  $|N_a - (q^{2a} + 1)| \leq \rho(F)q^a$ , where  $\rho(F)$  is the Picard number of the surface  $F$ .

Mordell’s hypothesis (2) on cubic surfaces is a special case of this corollary.

From Theorem 3 one easily obtains the following result:

**Theorem 4.** *Let  $k$  be a number field of finite degree; let  $F$  be a rational surface over the field  $k$ ; let  $G$  be the Galois group of a normal extension of the field  $k$  over which there exists a basis of the group  $C(F)$  of divisor classes of the surface  $F$ . The Hasse-Weil zeta-function of the surface  $F$  differs, by only a finite number of Euler factors, from the product  $\zeta(s, k)\zeta(s - 2, k)L(s, k)$ , where  $L(s, k)$  is the Artin  $L$ -function associated with the representation of the group  $G$  in the  $Z$ -module  $C(F)$ .*

Theorem 3 is derived by the standard method from the following fact.

**Lemma 2.** *Let  $A(V)$  be the group of cycle classes on an algebraic variety  $V$  with respect to numerical equivalence. Then for any two rational surfaces  $E, F$ , the group  $A(E \times F)$  is generated by classes of cycles of the form  $x \times y$ ,  $x \subset E$ ,  $y \subset F$ .*

**Proof.** We shall use the results and the technique of § 5 of Grothendieck's report <sup>(9)</sup>. First let  $E, F$  be minimal models. If one of the surfaces  $E, F$  is isomorphic to a quadric or to the projective plane, then the assertion of the lemma is contained in the corollary to Proposition 7 of Grothendieck <sup>(9)</sup>, p. 32). If  $F = F_n$ , then one can apply Proposition 7 directly to the decomposition  $X_2 = E \times F$ ,  $X_1 = E \times P^1$ ,  $X_0 = E \times Q$ , where  $P^1 \subset F$  is some fiber of the surface  $F_n$  with respect to the projection  $\pi$ , and  $Q \in P^1$  is a point. The set  $S_i \subset A(X_i)$  is composed of classes of cycles of the form

$x \times y$ , contained in  $A(X_i)$ . It is easy to see that (in Grothendieck's notation)  $v_i^*(S_i)$  generates the group  $A(X_i - X_{i-1})$  for  $i = 0, 1$ , where  $v_i : (X_i - X_{i-1}) \rightarrow X_i$  is the embedding morphism. It remains to verify this for  $i = 2$ . We have  $X_2 - X_1 \simeq E \times P^1 \times (b - Q)$ , where  $b$  is the base of the surface  $F$ . Since  $b - Q$  is an affine line, it follows from Proposition 6 and the corollary to Proposition 7 that  $A(X_2 - X_1)$  is generated by classes of cycles of the form  $x \times F$  and  $x \times b$ ,  $x \in E$ . Thus the conditions of Proposition 7 are fulfilled, whence the required assertion follows. Now assume that the lemma has been proved for the surfaces  $E, F$ ; let  $Q \in F$  be some point, and let  $F'$  be the image of  $F$  under the quadratic transformation with center  $Q$ . It is enough to prove that the lemma is then valid for the surfaces  $E, F'$ , because the assertion is symmetric with respect to  $E, F$ , and every surface is obtained from a minimal one by a finite number of quadratic transformations. Let  $P^1 \subset F'$  be the image of the point  $Q$ . Apply Proposition 7 to the partition  $X_1 = E \times F'$ ,  $X_0 = E \times P^1$ , choosing  $S_i$  as before. Again it is enough to verify that  $v_1^*(S_1)$  generates  $A(X_1 - X_0)$ . But

$$X_1 - X_0 = E \times (F' - P^1) \simeq E \times (F - Q).$$

By the induction hypothesis,  $A(E \times F)$  is generated by classes of cycles of the form  $x \times y$ ; the same is therefore true for  $A(E \times (F - Q))$ , for, by the exactness axiom <sup>(9)</sup>, p. 26), the restriction map

$$A(E \times F) \rightarrow A(E \times (F - Q))$$

is an epimorphism. The lemma is proved.

Applying this lemma to the case  $E = F$  and restricting ourselves to the consideration of two-dimensional cycles on  $E \times E$ , we easily obtain the Lefschetz formula, written in the report of A. Weil. From this formula, as applied to powers of the Frobenius endomorphism, Theorem 3 follows.

5. In conclusion, let us formulate several unsolved problems. The most important questions remain open: the existence and effective finding of a rational point on a surface, and also the description of all such points.

To solve these questions it is important to be able to describe forms of surfaces up to birational equivalence over the ground field, which leads to the necessity of studying Galois cohomology of the group of Cremona transformations. Apparently little is known about them. It would be interesting to know how close the analogy is with the cohomology of linear groups. Are there nontrivial birational forms of the plane over a finite field?

By considerations close to those used in this note, one can show that the Minkowski–Hasse principles are still valid for rational surfaces with Picard number  $\rho \leq 6$ . The known counterexamples pertain to the cases  $\rho = 7$  and  $\rho \geq 9$ , respectively. For what least value of  $\rho$  do such examples exist?

Attempts to generalize our results to rational varieties of higher dimensions run first of all into the insufficient study of the geometry of these varieties. We note in this connection that the classes of minimal rational surfaces correspond one-to-one to conjugacy classes of “maximal finite subgroups of Lie” of the group of birational transformations of the plane (see <sup>(10)</sup>). Does this correspondence persist for three-dimensional varieties? A positive answer to this question would make it possible to list the minimal rational varieties of dimension 3, since the maximal subgroups of the Cremona transformations of space have been described in the classical theory.

Steklov Mathematical Institute  
Academy of Sciences of the USSR

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