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Abstract

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RELATIONS BETWEEN THE RADII OF THE INSCRIBED AND CIRCUMSCRIBED SPHERES OF A CLOSED CONVEX SURFACE

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Let $u = u(x_1, x_2, \dots, x_n)$ $(n = 2, 3, \dots)$ be a continuous symmetric function, strictly monotonically increasing in each argument, defined in the domain $x_1 > 0, x_2 > 0, \dots, x_n > 0$, and let $u(1, 1, \dots, 1) = 0$.

We define the classes of n-dimensional surfaces Φ_u and Φ^u in $(n+1)$-dimensional Euclidean space E^{n+1}. We shall say that a closed convex surface $\Phi \in \Phi_u$ if its principal radii of curvature $R_1, R_2, \dots, R_n$ at every point satisfy the relation $u(R_1, R_2, \dots, R_n) \leqslant 0$, and $\Phi \in \Phi^u$ if $u(R_1, R_2, \dots, R_n) \geqslant 0$.

To each closed convex surface Φ we assign the ordered pair of numbers (R, r), where the first number is the radius of the smallest sphere circumscribed about Φ, and the second is the radius of the largest sphere inscribed in Φ. Denote this mapping by $F(\Phi)$. In this paper the problem is solved of finding the sets of images of the surfaces of the classes Φ_u and Φ^u under the mapping $F(\Phi)$.

Denote by $v = v(s)$ the function defined by the equation $u(v, s, \dots, s) = 0$, and by (τ, T) the largest interval on which the function $v = v(s)$ is defined. In the present note, for simplicity, we shall assume that the function $v = v(s)$ satisfies the Lipschitz condition.

On the intervals $(\tau, 1)$ and $(1, T)$ define the function

$$t(q) = \begin{cases} \tau \dfrac{\cos^2 \alpha}{\sin \alpha} + \displaystyle\int_0^\alpha \cos\psi \cdot v(p)\, d\psi, & \text{if } \tau < q < 1, \\ \displaystyle\int_0^\alpha \cos\psi \cdot v(p)\, d\psi, & \text{if } 1 < q < T, \end{cases}$$

where

$$\alpha = \begin{cases} \arccos \exp\left[-\displaystyle\int_\tau^q \frac{ds}{v(s) - s}\right], & \text{if } \tau < q < 1, \\ \arccos \exp\left[-\displaystyle\int_q^T \frac{ds}{s - v(s)}\right], & \text{if } 1 < q < T, \end{cases}$$

and $p = p(\psi, q)$ is determined by the relations:

$$\int_p^q \frac{ds}{v(s)-s} = -\ln\cos\psi, \quad \text{if } \tau < q < 1, \quad 0 \leqslant \psi < \alpha(q);$$

$$\int_q^p \frac{ds}{s-v(s)} = -\ln\cos\psi, \quad \text{if } 1 < q < T, \quad 0 \leqslant \psi < \alpha(q).$$

It can be shown that

$$\lim_{q\to 1} t(q) = 1.$$

In the plane (R, r) define two sets Δ_u and Δ^u.
The set Δ_u is given by the inequalities:

$$\begin{array}{ll} r \le R & \text{for } 0 < r \le \tau; \\ r \le R < t(r) & \text{for } \tau < r < 1; \\ r = R & \text{for } r = 1. \end{array}$$

The set Δ^u is given by the inequalities:

$$\begin{array}{ll} r = R & \text{for } R = 1; \\ t(R) < r \le R & \text{for } 1 < R < T; \\ 0 < r \le R & \text{for } T \le R. \end{array}$$

With the notation introduced, the following theorems hold:

Theorem 1. $F(\Phi)$ maps the class of surfaces Φ_u onto the set Δ_u.

Theorem 2. $F(\Phi)$ maps the class of surfaces Φ^u onto the set Δ^u.

Let us consider some special cases.

1°. For $u = x_1 x_2 \cdots x_n - 1$, Φ_u (Φ^u) consists of surfaces whose Gaussian curvature at each point is not less (not greater) than one. In this case the function

$$t(q) = \int_0^\alpha \frac{\cos^n\psi\, d\psi}{(q^n - 1 + \cos^n\psi)^{(n-1)/n}},$$

where

$$\alpha = \begin{cases} \arccos(1-q^n)^{1/n}, & \text{for } 0 < q < 1; \\ \dfrac{\pi}{2}, & \text{for } 1 < q < \infty. \end{cases}$$

Since the diameter of the circumscribed ball is not less than the diameter of the surface, and the diameter of the inscribed ball is not greater than the width of the surface, it follows, for $u = x_1 x_2 - 1$, from Theorems 1 and 2, that the theorems of Blaschke and Bonnesen follow (see ([1]), pp. 134, 144).

2°. For $u = x_1 + x_2 + \dots + x_n - n$, Φ_u (Φ^u) consists of surfaces for which the arithmetic mean of the principal radii of curvature at each point is not greater (not less) than one. For $n = 2$ the class Φ_u (Φ^u) coincides with the class of surfaces whose Gaussian curvature is not less (not greater) than the mean curvature. In this case the function

$$t(q) = \left[1 - (1-q)^{2/n}\right]^{1/n} + (n-1)(1-q) \int_0^\alpha \frac{d\psi}{\cos^{n-1}\psi},$$

where

$$\alpha = \begin{cases} \arccos(1-q)^{1/n}, & \text{for } 0 < q < 1; \\ \arccos\left[(n-1)(q-1)\right]^{1/n}, & \text{for } 1 < q < \dfrac{n}{n-1}. \end{cases}$$

From this formula one can find the greatest value of the function $t(q)$ for $0 < q < 1$. It is equal to $1/\sin\beta$, where $\beta > 0$ is found from the equation

$$\int_0^\beta \frac{d\psi}{\cos^{n-1}\psi} = \frac{1}{(n-1)\cos^{n-2}\beta \sin\beta}.$$

Consequently, the following holds.

Theorem 3. If $u = x_1 + x_2 + \dots + x_n - n$, then every surface $\Phi \in \Phi_u$ is contained in a ball of radius $\dfrac{1}{\sin\beta}$. The estimate obtained cannot be improved $\left(\text{for } n = 2,\ \dfrac{1}{\sin\beta} \simeq 1.20\right)$.

3°. For $u = \dfrac{2x_1 x_2}{x_1 + x_2} - 1$, $\Phi_n(\Phi^u)$ consists of surfaces whose mean curvature at each point is not less than (not greater than) one. In this case the function

$$t(q) = \begin{cases} \dfrac{1}{2(2q-1)} + \dfrac{1}{2} \displaystyle\int_0^{2q-1} \sqrt{\frac{1-z^2}{(2q-1)^2 - z^2}}\, dz, & \text{for } \dfrac{1}{2} < q < 1, \\ \dfrac{1}{2} + \dfrac{1}{2} \displaystyle\int_0^1 \sqrt{\frac{1-z^2}{(2q-1)^2 - z^2}}\, dz, & \text{for } 1 < q < \infty. \end{cases}$$

From this formula it is seen that the exact lower bound of $t(q)$ for $q > 1$ is equal to $1/2$. Consequently, the following holds.

Theorem 4. *If the mean curvature of a convex surface does not exceed one at every point, then it contains a ball of radius* $1/2$. *This estimate cannot be improved.*

We now give a brief exposition of the idea of the proof of Theorem 1. For simplicity we restrict ourselves to the case $\tau = 0$. Let $\Phi \in \Phi_u$, and let O be the center of the largest ball σ inscribed in the surface Φ. It is obvious that on Φ there is a point A at a distance R from O, where R is the radius of the smallest ball circumscribed about Φ. Introduce coordinates on the line OA so that the points O and A have coordinates 0 and R. Denote by $Q(x)$ the n-dimensional plane passing through the point x of the line OA perpendicular to the latter. On the segment $[0, R]$ define the function $f_0(x)$, equal to the distance from the point x $(0 \leq x \leq R)$ of the line OA to the intersection of Φ with $Q(x)$. It is not difficult to show, using the maximality of the inscribed ball σ, that on the segment $[0, r]$ (r is the radius of the ball σ) there exists a number x_0 satisfying the equation

$$f_0(x_0) = \sqrt{r^2 - x_0^2}.$$

Define a new function on the segment $[-R, R]$:

$$f(x) = \begin{cases} \sqrt{r^2 - x^2}, & \text{if } 0 \leq |x| \leq x_0, \\ f_0(|x|), & \text{if } x_0 \leq |x| \leq R. \end{cases}$$

In each plane $Q(x)$ $(-R \leq x \leq R)$ construct an $(n-1)$-dimensional sphere of radius $f(x)$ with center on the line OA. The family of these spheres forms a certain n-dimensional surface Ψ. It can be shown that Ψ is a closed convex surface of revolution and that at those of its points where the principal radii of curvature $R_1, R_2, \ldots, R_n$ exist (and they exist almost at every point of Ψ, see ([2]), p. 3), the inequality

$$u(R_1, R_2, \ldots, R_n) \leq 0$$

holds. Now it is not difficult to prove that almost everywhere on the segment $[-R, R]$ the function $y = f(x)$ satisfies the differential inequality

$$u\left[-\frac{(1+y'^2)^{3/2}}{y''},\ y(1+y'^2)^{1/2},\ \ldots,\ y(1+y'^2)^{1/2}\right] \leq 0.$$

It is further proved that the solution $y = g(x)$ of the equation

$$u\left[-\frac{(1+y'^2)^{3/2}}{y''},\ y(1+y'^2)^{1/2},\ \ldots,\ y(1+y'^2)^{1/2}\right] = 0$$

with the initial conditions $g(0) = r < 1$, $g'(0) = 0$, is defined and positive on the interval $(-t(r), t(r))$, and $g(R) > f(R) = 0$. Hence one may derive that $R < t(r)$. This completes the proof of Theorem 1. Theorem 2 is proved analogously.

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Note: Figure translations are in progress. See original paper for figures.