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S. A. EREMIN, A. F. SHADROV

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Abstract

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MATHEMATICS

S. A. EREMIN, A. F. SHADROV

ON COMPLETE SYSTEMS AND BASES IN SPACES OF FUNCTIONS ANALYTIC IN HARTOGS DOMAINS

(Presented by Academician V. I. Smirnov, 2 VIII 1962)

In the present work we develop the multiplicative method for constructing complete systems and bases, the idea of which was given in [1]. The method is applied here chiefly to the construction of complete systems and bases in spaces of functions analytic in Hartogs domains.

Let D_1, \dots, D_q be finite domains, respectively, of the spaces of the complex variables $z_1, \dots, z_k, \dots, z_m, \dots, z_n$, regularly (semicontinuously) extendable to the domains T_1, \dots, T_q ; $D_1 \Subset T_1, \dots, D_q \Subset T_q$; $D = D_1 \times \dots \times D_q$ is the topological product of the domains D_1, \dots, D_q ; $E(G)$ is the space of all functions analytic in the domain G ; convergence in $E(G)$ is defined as uniform convergence on every closed set of points of the domain G .

Theorem 1. *For every domain D_j there exists a system of functions complete in $E(D_j)$.*

Proof. By virtue of the regular extendability of D_j to T_j , all functions analytic in D_j are uniformly approximated inside D_j by functions analytic in T_j [2]. On the other hand, for D_j there exists a closed orthonormal system $\{\psi_{\nu_j}\}$ of functions analytic in D_j [3, 4], and therefore every function $F \in E(T_j)$ is representable inside D_j by uniformly convergent series of the form $\sum c_{\nu_j}^{(j)} \psi_{\nu_j}$. Consequently, the system $\{\psi_{\nu_j}\}$ is complete in $E(D_j)$.

Theorem 2. *In order that the system*

$$\{\varphi_{\nu_1}(z_1, \dots, z_k) \cdots \varphi_{\nu_q}(z_m, \dots, z_n)\} \quad (1)$$

be complete in $E(D)$, it is necessary and sufficient that each system

$$\{\varphi_{\nu_j}\} \quad (j = 1, \dots, q) \quad (2)$$

be complete in the corresponding space $E(D_j)$.

Proof. From the completeness of the system (1), the completeness of the systems (2) follows directly. By simple calculations one is convinced that, to prove the converse assertion, it is sufficient to prove it for the case of complete orthonormal systems $\{\psi_{\nu_j}\}$. Any function $F \in E(D)$, as a function of z_1, \dots, z_k , is expanded inside D_1 into uniformly convergent series (with respect to z_1, \dots, z_k) in $\{\psi_{\nu_1}\}$ with coefficients that are analytic functions of the variables z_{k+1}, \dots, z_n . These coefficients, in turn, are expanded into uniformly convergent series in $\{\psi_{\nu_2}\}$, and so on.

Consequently, F is expanded inside D into uniformly convergent series in the functions of the system $\{\psi_{\nu_1} \cdots \psi_{\nu_q}\}$, and the theorem is proved.

Theorem 3. In order that system (1) be a basis of the space $E(D)$, it is necessary and sufficient that each system (2) be a basis of the corresponding space $E(D_j)$.

By virtue of Theorem 2, the proof of Theorem 3 is carried out by arguments analogous to those given in the proof of the corresponding theorem [1].

Let $G\{|z_1| < R_G(z_2, \dots, z_n), (z_2, \dots, z_n) \in H_G\}$ be a complete Hartogs domain with plane of symmetry $z_1 = 0$, where H_G is an arbitrary domain.

Theorem 4. If the system

$$\{f_{k_2, \dots, k_n}(z_2, \dots, z_n)\} \quad (3)$$

is complete in $E(H_G)$, then the system

$$\{z_1^{k_1} f_{k_2, \dots, k_n}(z_2, \dots, z_n)\} \quad (4)$$

is complete in $E(G)$.

Proof. Let system (3) be complete in $E(H_G)$. Since for any $f(z_1, \dots, z_n) \in E(G)$ in each domain $G_0 \Subset G$ the Hartogs representation is valid,

$$f(z_1, \dots, z_n) = \lim_{m_1 \rightarrow \infty} \sum_{k_1=0}^{m_1} z_1^{k_1} f_{k_1}(z_2, \dots, z_n),$$

then uniformly in $G_0 \Subset G$

$$\begin{aligned} f(z_1, \dots, z_n) &= \lim_{m_1 \rightarrow \infty} \sum_{k_1=0}^{m_1} z_1^{k_1} \left(\lim_{m_2, \dots, m_n \rightarrow \infty} \sum_{k_2, \dots, k_n=0}^{m_2, \dots, m_n} c_{k_2, \dots, k_n}^{(k_1)} f_{k_2, \dots, k_n}(z_2, \dots, z_n) \right) \\ &= \lim_{m_1, \dots, m_n \rightarrow \infty} \sum_{k_1, \dots, k_n=0}^{m_1, \dots, m_n} c_{k_2, \dots, k_n}^{(k_1)} z_1^{k_1} f_{k_2, \dots, k_n}(z_2, \dots, z_n). \end{aligned}$$

Consequently, system (4) is complete in $E(G)$.

Theorem 5. If system (3) is a basis of the space $E(H_G)$, then system (4) is a basis of the space $E(G)$.

Proof. Substituting, in the Hartogs expansion of $f(z_1, \dots, z_n) \in E(G)$, the expansions of the functions $f_{k_1}(z_2, \dots, z_n)$ with respect to the basis (3), we obtain the series

$$\begin{aligned} f(z_1, \dots, z_n) &= \sum_{k_1=0}^{\infty} z_1^{k_1} \left(\sum_{k_2, \dots, k_n} c_{k_2, \dots, k_n}^{(k_1)} f_{k_2, \dots, k_n}(z_2, \dots, z_n) \right) \\ &= \sum_{k_1, \dots, k_n=0}^{\infty} c_{k_2, \dots, k_n}^{(k_1)} z_1^{k_1} f_{k_2, \dots, k_n}(z_2, \dots, z_n), \end{aligned}$$

which converges uniformly in each domain $G_0 \Subset G$. Consequently, system (4) is a basis of $E(G)$.

Theorems 4 and 5 make it possible, for every known criterion for completeness or for being a basis of a system in the case of k complex variables, to establish the validity of the corresponding criterion in the case of $(k+1)$ variables.

For example, a number of criteria for completeness and for being a basis of a system in the case when G is a complete Hartogs domain in the space of three variables are established from known theorems ⁽⁵⁾, p. 70, Theorems 1, 2, 3; p. 74, Theorems 4, 5; p. 75, Theorem 8; p. 78; ⁽⁶⁾, Theorems 2-4; ⁽⁷⁾, Theorems 1-5; ⁽⁸⁾, Theorems 2-5). Here the theorems corresponding to Theorem 8 on p. 78 in ⁽⁵⁾, to Theorems 2-4 in ⁽⁶⁾, to Theorems 1-5 in ⁽⁷⁾, and to Theorems 2-5 in ⁽⁸⁾, are established for the case when H_G is a bounded complete bicircular domain, since it is known that the space

functions analytic in a bounded complete bicircular domain is isomorphic to the space of functions analytic in a bicylinder ([9]).

Let

$$G\{|z_{n+1}| < R_G(z_1, \dots, z_n), (z_1, \dots, z_n) \in H_G\},$$

where $H_G = D_1 \times \dots \times D_q$.

Theorem 6. If each system

$$\{\varphi_{\nu_1}(z_1, \dots, z_k)\}, \dots, \{\varphi_{\nu_q}(z_m, \dots, z_n)\} \quad (5)$$

is complete respectively in $E(D_1), \dots, E(D_q)$, then the system

$$\{z_{n+1}^{k_{n+1}} \varphi_{\nu_1}(z_1, \dots, z_k) \cdots \varphi_{\nu_q}(z_m, \dots, z_n)\} \quad (6)$$

is complete in $E(G)$.

Theorem 7. If the systems (5) are bases respectively of the spaces $E(D_1), \dots, E(D_q)$, then the system (6) is a basis of $E(G)$.

The validity of Theorems 6 and 7 follows respectively from Theorems 2 and 4 and Theorems 3 and 5.

Let us have a sequence of complete Hartogs domains

$$\begin{aligned} &G_2\{|z_2| < R_{G_2}(z_1), z_1 \in G_1\}, \\ &G_3\{|z_3| < R_{G_3}(z_1, z_2), (z_1, z_2) \in G_2\}, \\ &\dots\dots\dots \\ &G_n\{|z_n| < R_{G_n}(z_1, \dots, z_{n-1}), (z_1, \dots, z_{n-1}) \in G_{n-1}\} \end{aligned}$$

respectively with symmetry planes $z_2 = 0, \dots, z_n = 0$, where G_1 is an arbitrary domain of the space of the complex variable z_1 .

Theorem 8. If the system $\{f_{k_1}(z_1)\}$ is complete in $E(G_1)$, then the system $\{z_n^{k_n}, \dots, z_2^{k_2} f_{k_1}(z_1)\}$ is complete in $E(G_n)$.

Theorem 9. If the system $\{f_{k_1}(z_1)\}$ is a basis of the space $E(G_1)$, then the system $\{z_n^{k_n} \dots z_2^{k_2} f_{k_1}(z_1)\}$ is a basis of $E(G_n)$.

The validity of Theorems 8 and 9 is established by $(n - 1)$ -fold application respectively of Theorems 4 and 5.

With the aid of Theorems 8 and 9, for every known theorem on the completeness or basicity of a system in $E(G_1)$ one can establish the corresponding theorem for the space of functions analytic in a Hartogs domain of any number of variables.

Analogously to Theorems 8 and 9, with the aid of Theorems 4 and 5, for a sequence of Hartogs domains

$$\begin{aligned} &G_{m+1}\{|z_{m+1}| < R_{G_{m+1}}(z_1, \dots, z_m), (z_1, \dots, z_m) \in G_m\}, \\ &\dots\dots\dots \\ &G_n\{|z_n| < R_{G_n}(z_1, \dots, z_{n-1}), (z_1, \dots, z_{n-1}) \in G_{n-1}\}, \end{aligned}$$

where G_m is any domain of the space of complex variables z_1, \dots, z_m , the following theorems are established:

Theorem 10. If the system $\{f_{k_1, \dots, k_m}(z_1, \dots, z_m)\}$ is complete in $E(G_m)$, then the system $\{z_n^{k_n} \dots z_{m+1}^{k_{m+1}} f_{k_1, \dots, k_m}(z_1, \dots, z_m)\}$ is complete in $E(G_n)$.

Theorem 11. If the system $\{f_{k_1, \dots, k_m}(z_1, \dots, z_m)\}$ is a basis of the space $E(G_m)$, then the system $\{z_n^{k_n} \dots z_{m+1}^{k_{m+1}} f_{k_1, \dots, k_m}(z_1, \dots, z_m)\}$ is a basis of $E(G_n)$.

Theorems 10 and 11 allow, for every known criterion of completeness or basicity of a system in the space $E(G_m)$ of m complex

variables, establish the corresponding criterion for the space of functions analytic in a complete Hartogs domain of any number of variables greater than m .

For example, a number of criteria for the completeness and basis property of a system in $E(G_n)$ are established by the theorems indicated above (5⁻⁸).

Kuibyshev Civil Engineering Institute
named after A. I. Mikoyan

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