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1963

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Abstract

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ON THE QUESTION OF JUSTIFYING THE CONVERGENCE OF DIFFERENCE SCHEMES

(Presented by Academician I. G. Petrovsky, 22 I 1963)

A general result on the convergence of analogues of difference methods for evolution equations in Banach spaces was obtained by Lax (^{1}) (see (^{2})). From the point of view of applications, this result has shortcomings: 1) the consistency condition for the original and approximate operators requires convergence of the approximate operators to the original one on an everywhere dense set of solutions of the evolution equation; 2) the approximate equation is considered in the same Banach space as the basic one; in reality, when difference methods are applied, the approximate equation is solved in the class of functions defined only at the nodes of the mesh.

In the present article, Lax' s theorem is developed in the following directions: it is proved that the consistency condition may be checked only on a certain set of initial values of solutions, the structure of which is described. The approximate equations are considered in another normed space. The article considers not only homogeneous equations with coefficients constant in t, but also inhomogeneous equations with variable coefficients, as well as some nonlinear equations.

Consider on the segment $[0, T]$ the differential equation

$$du/dt + A(t)u = f(t), \tag{1}$$

where $A(t)$, for each $t \in [0, T]$, is a linear unbounded closed operator acting in a Banach space E, with dense domain of definition $D(A)$ independent of t; $f(t)$ is a given function with values in E; $u(t)$ is the unknown function satisfying the initial condition

$$u(0) = u_0. \tag{2}$$

In works ($^{3-5}$), under certain assumptions, the existence of a solution of problem (1)–(2) was established.

Suppose that the space E contains a narrower Banach space E_1 such that $\|v\|_E \leq C_1\|v\|_{E_1}$ and $D[A^\alpha(t)] \subset E_1$ for some $\alpha > 0$, and moreover

$$\|v\|_{E_1} \leq C_2\|A^\alpha(t)v\|_E \quad (v \in D[A^\alpha(t)]). \tag{3}$$

Let there exist a sequence of bounded operators $A_n(t)$ acting in E_1 such that the operators $A_n(t)A^{-1}(t)$ are uniformly bounded in E_1 for each n, and the following consistency condition holds:

C. For any $v \in E$, as $n \to \infty$,

$$\sup_{t\in[0,T]} \|[A_n(t) - A(t)]A^{-1-\alpha}(t)v\|_{E_1} \to 0.$$

Usually the operators $A_n(t)$ are "degenerate." Let the subspace L_n of the space E_1, consisting of all elements of E_1 on which the operator $A_n(t)$ vanishes, be independent of t. Denote by S_n the factor space of E_1 by the subspace L_n: $S_n = E_1/L_n$. We shall assume that the norm in the space S_n is introduced so that, for $v \in S_n$,

$$\|\bar{v}\|_{S_n} \leq \inf_{v\in\bar{v}} \|v\|_{E_1}.$$

The operator $A_n(t)$ naturally generates in S_n the operator $\overline{A_n(t)}$ by the formula

$$\bar{A}_n(t)\bar{v} = \overline{A_n(t)v}.$$

We construct in the space S_n a finite-difference analogue of problem (1)–(2)

$$\frac{\bar{v}_{l+1} - \bar{v}_l}{\Delta_n t} + \bar{A}_n(l\Delta_n t)\bar{v}_l = \bar{f}_l; \tag{4}$$

$$\bar{v}_0 = \bar{u}_0. \tag{5}$$

The paper considers the question of under what conditions the solution of the finite-difference problem (4)–(5) converges to the solution of problem (1)–(2).

§ 1. Let us first consider the homogeneous equation with constant operator. This equation has a solution satisfying the initial condition $u(0) = u_0 \in D(A)$; if the operator $-A$ is the infinitesimal generator of a strongly continuous semigroup $U(t)$ of bounded operators, then the solution of the problem is given by the formula $u(t) = U(t)u_0$, and moreover, if $u_0 \in D(A^k)$, then $u(t) \in D(A^k)$ (see ([3])).

In this case we shall assume that $\alpha = k$, where k is a positive integer. Then, if $u_0 \in D(A^k)$, then $u(t) \in E_1$, and, consequently, $u(t)$ is matched by some element $\bar{u}(t)$ of the space S_n.

For the corresponding finite-difference problem the solution in the space S_n can be written in the form $\bar{v}_l = (1 - \Delta_n t\bar{A}_n)^l \bar{u}_0$.

Under these conditions the following assertion holds:

Theorem 1. Let $f(t) \equiv 0$ and $A(t) \equiv A$. Suppose that for $\alpha = k$ the consistency condition C and the stability condition

$$. \quad \left\|(1-\Delta_n t\bar{A}_n)^l\right\|_{S_n} \leq M \quad \text{for all } l\Delta_n t \in [0,T]$$

are satisfied.

Then, if $u_0 \in D(A^k)$, the solution of the finite-difference problem converges to the solution of problem (1)–(2) in the sense that, as $\Delta_n t \to 0$,

$$\left\|\overline{u(t)} - \bar{v}_{k_n}\right\|_{S_n} \to 0$$

uniformly in t on $[0,T]$, where $k_n = [t/\Delta_n t]$.

The consistency condition for the case under consideration is equivalent to the following:

$$\|A_{nu}0 - Au_0\|_{E_1} \to 0 \quad \text{as } n \to \infty \text{ for any } u_0 \in D(A^{k+1}).$$

§ 2. The homogeneous equation with variable operator was studied in the works (4,5). In (4) it is shown that the solution of such an equation under the initial condition $u(0) = u_0 \in D(A)$ exists and is unique if, for every $t \in [0,T]$,

$$\left\|[\lambda I + A(t)]^{-1}\right\|_E \leq \frac{1}{1+\lambda} \quad (\lambda > -1)$$

and the operator $C(t) = A(t)dA^{-1}(t)/dt$ is bounded and strongly continuous in t.

If, in addition, the operator $A(t)A^{-1}(0)$ has a second strongly continuous derivative with respect to t, then the theorem holds:

Theorem 2. Let, for $\alpha = 1$, the consistency condition C and the stability condition

$$. \quad \left\|1-\Delta_n t\bar{A}_n(t)\right\|_{S_n} \leq 1 + C\Delta_n t \quad \text{for all } t \in [0,T].$$

be satisfied.

If $f(t) \equiv 0$ and $u_0 \in D(A)$, then the solution of the finite-difference problem converges to the solution of problem (1)–(2) in the sense that, as $\Delta_n t \to 0$,

$$\left\|\overline{u(t)} - \bar{v}_{k_n}\right\|_{S_n} \to 0$$

uniformly in t on $[0,T]$, where $k_n = [t/\Delta_n t]$.

From the results of (^{5}) it follows that the solution of the homogeneous problem (1)–(2) exists under the following assumptions:

$1'$. The operator $A(t)$, for any λ with $\operatorname{Re}\lambda \geq 0$, has a resolvent $(A(t)+\lambda I)^{-1}$, whose norm satisfies the inequality

$$\left\|(A(t)+\lambda I)^{-1}\right\|_E \leq \frac{C_1}{|\lambda|+1} \quad (0 \leq t \leq T).$$

$2'$. For any $t, \tau, s \in [0, T]$

$$\left\|[A(t)-A(\tau)]A^{-1}(s)\right\|_{E_1} \leq C|t-\tau|^{\gamma}$$

for some $\gamma \in (0, 1]$.

Suppose, moreover, that the condition is satisfied:

3. The operator $A(t)A^{-1}(0)$ is strongly differentiable with respect to t, and the operator $A'(t)A^{-1}(t)$ satisfies a Hölder condition; then Theorem 2 remains valid. Moreover, Theorem 2 admits the following strengthening.

Theorem $2'$. *Suppose that for* $\alpha = 1$ *the consistency condition* C *and the stability condition* Y *are satisfied. Then, for any* u_0 *from the closure of* D *in the space* E_1*, the solution of the finite-difference homogeneous problem converges to the solution of the homogeneous problem (1)–(2) uniformly in* t *on every interval* $[a, T]$*,* $a > 0$.

§ 3. For the nonhomogeneous problem (1)–(2), in (5) the existence of a solution is proved if the operator A satisfies conditions $1'$, $2'$, and the function $f(t)$ satisfies the Hölder condition

$$\|f(t)-f(s)\|_E \leq C|t-s|^{\rho} \quad (C, \rho > 0). \tag{6}$$

Let now $D[A^{\gamma_1}(t)] \subset E_1$, for some $\gamma_1 \in (0, \min(\gamma, \rho))$ $(\alpha = \gamma_1)$. The solution of problem (1)–(2) for $t \geq 0$ and $u_0 \in D[A^{\gamma_1}(0)]$ belongs to the space E_1, and consequently $\overline{u(t)} \in S_n$.

Suppose that in the space E_1 one can introduce a seminorm $\| \; \|_n$ such that:

1^0. $\|v\|_n = 0$ if and only if $v \in L_n$.

2^0. $\|v\|_n \leq C\|v\|_{E_1}$ (C does not depend on n).

3^0. $\|u\|_E - \varepsilon_n\|v\|_{E_1} \leq \|v\|_n \leq \|v\|_E + \varepsilon_n\|v\|_{E_1}$ ($\varepsilon_n \to 0$ as $n \to \infty$).

Then in the space S_n one can introduce a norm $\| \; \|_n$: $\|\overline{v}\|_n = \|v\|_n$, where $v \in \overline{v}$.

Under assumptions $1'$, $2'$, and (6), the theorem holds:

Theorem 3. *Suppose that for* $\alpha = \gamma_1$ *the consistency condition* C *and the stability condition*

$$\text{Y.} \quad \|1 - \Delta_{nt}\overline{A}_n(t)\|_n \le 1 + C\Delta_{nt} \quad \text{for } t \in [0,T].$$

are satisfied.

Suppose the function $f(t)$ *is continuous in the norm* $\| \ \|_{E_1}$.

Then, if $u_0 \in D[A^{\gamma_1}(0)]$, *the solution of problem (4)–(5) converges to the solution of problem (1)–(2) in the sense that, as* $\Delta_{nt} \to 0$,

$$\|\overline{u(t)} - \overline{v}_{kn}\|_n \to 0$$

uniformly in $t \in [0,T]$.

§ 4. We now consider a nonlinear equation of the form

$$\frac{du}{dt} + A(t)u = f(t,u); \tag{7}$$

here we shall assume that the operator $A(t)$ satisfies conditions $1'$, $2'$.

Let the function $f(t, A^{-\alpha}v)$, for some $\alpha \in [0,1)$, act in E for every $t \in [0,T]$, and suppose that for any $t, \tau \in [0,T]$ and $v, w \in E$ with $\|v\|_E, \|w\|_E \le R$ (R is some positive number), the inequality

$$\|f(t, A^{-\alpha}v) - f(\tau, A^{-\alpha}w)\|_E \le C_R(|t-\tau|^\lambda + \|v-w\|_E^\mu), \tag{8}$$

holds, where λ, μ are some positive numbers not exceeding 1.

Under these assumptions, existence is proved (see (5)) on $[0, t_0] \subset [0,T]$ of a solution of equation (7) satisfying the initial condition

$$u(0) = u_0, \tag{9}$$

$u_0 \in D[A_0^\beta]$ for some $\beta > \alpha$.

Consider again a space $E_1 \subset E$ such that $D[A^{\gamma_1}(t)] \subset E_1$ and

$$\|v\|_{E_1} \le C_2\|A^{\gamma_1}(t)v\|_E \quad (v \in D[A^{\gamma_1}(t)])$$

for some $\gamma_1 \in (0, \min(\gamma, \lambda, \mu))$, and construct, as in the linear case, the space S_n. We shall assume–

assume that the norm in S_n is introduced by the formula

$$\|\bar{v}\|_{S_n} = \inf_{v \in \bar{v}} \|v\|_{E_1}.$$

Let the function $f(t, v)$ be such that if $v \in E_1$, then $f(t, v)$ also belongs to E_1, and for any $v \in \bar{v}$ ($\bar{v} \in S_n$), $f(t, v)$ corresponds to one and the same element of the space S_n.

Consider in S_n the finite-difference problem for (7)–(9)

$$\frac{\bar{v}_{k+1} - \bar{v}_k}{\Delta_n t} + \bar{A}_n \bar{v}_k = \bar{f}_k \quad (\bar{f}_k = \overline{f(t_k, v_k)}), \tag{10}$$

$$\bar{v}_0 = \bar{u}_0. \tag{11}$$

Under the indicated assumptions the following theorem holds:

Theorem 4. Let, for $a = \gamma_1$, the consistency condition C and the stability condition

$$. \ \|1 - \Delta_n t \bar{A}_n(t)\|_{S_n} \leq 1 + \bar{C}\Delta_n t \quad \text{for } t \in [0, t_0].$$

be satisfied.

Suppose that for any $t, \tau \in [0, T]$ and $v, w \in E_1$ with $\|v\|_{E_1}, \|w\|_{E_1} \leq R_1$ (R_1 such that $\|u(t)\|_{E_1} < R_1$ for $t \in [0, t_0]$), the function $f(t, v)$ satisfies the condition

$$\|f(t, v) - f(t, w)\|_{E_1} \leq C_{R_1}(|t - \tau|^\lambda + \|v + w\|_{E_1}).$$

In addition, let $f(t, u) \in D[A^{\gamma_2}(0)]$ and let $A^{\gamma_2}(0) f(t, u(t))$ be continuous in E with respect to t ($\gamma_2 > \gamma_1$, $\gamma_2 \in (0, \min(\gamma, \lambda, \mu))$).

Then, for $u_0 \in D[A^{\gamma_1}(0)]$, the solution of problem (10)–(11) converges to the solution of problem (7)–(9) in the sense that, as $\Delta_n t \to 0$,

$$\|\bar{u}(t) - \bar{v}_{k_n}\|_{S_n} \to 0,$$

uniformly with respect to $t \in [0, t_0]$.

§ 5. The results obtained are applied to the proof of convergence of stable explicit difference schemes for boundary-value problems for equations of the form

$$\frac{\partial u}{\partial t} + Lu = f,$$

where L is a strongly elliptic operator of order $2m$, with sufficiently smooth coefficients, given in a bounded domain G of s-dimensional space. The role of the space E is played by the space $L_p(G)$, and the role of E_1 by the space $C(G)$ or $C^\alpha(G)$. In this connection, the embedding theorems of S. L. Sobolev (^{6}), a priori estimates for solutions of elliptic equations (^{7}), and estimates of the resolvent and of fractional powers of elliptic operators (^{8,9}) are used.

The author expresses sincere gratitude to S. G. Krein, under whose supervision this work was carried out.

Received
19 I 1963

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