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Abstract

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MATHEMATICS

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BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS IN CONICAL DOMAINS

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We shall consider the elliptic equation

$$L(u) \equiv \sum_{\substack{0\le i_1+\cdots+i_n\le 2m-i\\ 0\le i\le 2m}} A^{i}_{i_1\ldots i_n}(x_1,\ldots,x_n)\,\frac{\partial^{2m-i}u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}} = f(x_1,\ldots,x_n) \qquad (1)$$

in a domain G, whose boundary Γ is infinitely differentiable except for one point O, which is the origin of coordinates. We assume that in a neighborhood of O, in local coordinates, the boundary surface is the cone

$$\sum_{i=0}^{n-1} x_i^2 = a_0 x_n^2.$$

It is allowed that for $n = 2$ the boundary is an angle equal to 2π. We seek a solution of (1) satisfying on Γ the conditions

$$B_j u = \sum_{\substack{0\le i_1+\cdots+i_n\le m_j-i\\ 0\le i\le m_j}} B^{(ij)}_{i_1\ldots i_n}(x_1,\ldots,x_n)\,\frac{\partial^{m_j-i}u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}} = \varphi_j(x_1,\ldots,x_n). \qquad (2)$$

The coefficients occurring in (1), (2), up to the boundary, will be regarded as infinitely differentiable functions of the polar coordinates $(r,\omega_1,\ldots,\omega_{n-1})$. Here

$$r = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2};$$

$\omega_1,\ldots,\omega_{n-1}$ is an arbitrary system of coordinates on the unit sphere without singularities inside G. The smoothness requirements on the boundary and on the coefficients can be weakened, but for simplicity we regard them as infinitely differentiable. Everything said below also applies to the case when the boundary has a finite number of conical points, and it is assumed that the operators B_j, everywhere except the point O, satisfy the Lopatinskii conditions.

We introduce definitions. By $W_2^{l,\alpha}$ we shall denote the closure, in the norm

$$\|u\|_G^{l,\alpha} = \left[\iint_G \frac{|u|^2}{r^{2l-\alpha}}\,dS + \iint_G r^{\alpha} \left(\sum \left|\frac{\partial^l u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\right|^2\right) dS\right]^{1/2},$$

of the manifold of infinitely differentiable functions in G; here r is the distance to the origin. Let, by definition,

$$\|u\|_G^{l,\alpha} = \left[\iint_G r^{\alpha} \sum_{i_1\cdots i_n} \left|\frac{\partial^l u}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}\right|^2 dS\right]^{1/2}.$$

By W_2^l we shall denote the space of S. L. Sobolev consisting of functions with square-summable generalized derivatives of order l. On the set of boundary functions we introduce the norm

$$\|\varphi\|_\Gamma^{l-\frac{1}{2},\alpha} = \inf \|v\|_G^{l,\alpha} + \int_\Gamma \frac{\varphi^2}{r^{2l-\alpha-1}}\,dS,$$

where the infimum is taken over all v, defined in G, such that $v|_\Gamma = \varphi$. Let also $\|\varphi\|^{l-1/2,\alpha} = \inf \|\tilde{v}\|_G^{l,\alpha}$. Consider the space $\Phi_{m,m_j}^{l,\alpha}$, consisting of vectors of the form $\psi = [f, \varphi_j]$, whose norm is defined as follows:

$$\|\psi\|^{l,\alpha} = \|\tilde{f}\|_G^{l-2m,\alpha} + \|f\|_{L_2(G)} + \sum_{j=1}^{m} \left(\|\widetilde{\varphi}_j\|^{l-m_j-1/2,\alpha} + \|\varphi_j\|_{L_2(\Gamma)}\right)$$

$$(l \geq \max m_j,\ 2m).$$

We shall say that problem (1), (2) is **normally solvable** in $\Phi_{m,m_j}^{l,\alpha}$ if there exists a space $\widetilde{\Phi} \in \Phi_{m,m_j}^{l,\alpha}$ such that the factor space $\Phi_{m,m_j}^{l,\alpha}/\widetilde{\Phi}$ is finite-dimensional and, for every $\psi \in \widetilde{\Phi}$, there is a solution of problem (1), (2) such that the estimate

$$\|\tilde{u}\|_{W_2^{l,\alpha}} \ll \|\psi\|^{l,\alpha} + \|u\|_{L_2}, \tag{3}$$

holds, while the dimension of the space of solutions of the homogeneous problem (1), (2) is finite.

Under the conditions imposed above, the following theorem holds:

Theorem 1. *There exists a meromorphic function $K(z)$, depending only on the values of the coefficients $A^{(0)}, B_j^{(0)}$ at the point O and on the opening of the cone, such that in every strip $|\operatorname{Im} z| < c_1$ it has a finite number of poles. If n is odd and on the line $\operatorname{Im} z = l - \alpha/2 - n/2$ there are no poles of $K(z)$, then problem (1), (2) is normally solvable in the space $\Phi_{m,m_j}^{l,\alpha}$. If n is even, on the line $\operatorname{Im} z = l - \alpha/2 - n/2$ there are no poles of $K(z)$, and the point $i(l-(\alpha+n+2)/2)$ is not a pole of $K(z)$, then (1), (2) is normally solvable in the space $\Phi_{m,m_j}^{l,\alpha}$.*

Under the conditions stated above, for the solution of problem (1), (2) belonging to $W_2^{0,\alpha}$, the following asymptotic expansion holds:

$$u = \sum_{j=1}^{N} \sum_{k=0}^{k_j} (\ln r)^k r^{-i\lambda_j} \Phi_j(\omega_1, \dots, \omega_{n-1}) +$$

$$\sum_{\substack{0 \le k \le \bar{k}_p \\ 0 \le k_1 + \dots + k_n \le p}} c_k (\ln r)^k x_1^{k_1}, \dots, x_n^{k_n} + w(x_1, \dots, x_n), \tag{4}$$

where λ_j are poles of $K(z)$ lying above the line $\operatorname{Im} z = -\alpha/2 + n/2$; $(k_j - 1)$ is the multiplicity of λ_j; $\Phi_j(\omega_1, \dots, \omega_{n-1})$ are certain infinitely differentiable functions; $\bar{k}_p = 0$ if ip is not a pole of $K(z)$, and if ip is a pole of $K(z)$, then $\bar{k}_p$ is the number smaller by one than its multiplicity. The function $w(x_1, \dots, x_n)$ satisfies the condition $w = o\left(r^{\max|\operatorname{Im} \lambda_j|}\right)$. Equality (4) may be normally differentiated an arbitrary number of times.

In the formulation of Theorem 1 it was required that the point $i(l-(\alpha+n+2)/2)$ not be a pole of $K(z)$. The assertion of the theorem remains true if $i(l-(\alpha+n+2)/2)$ is a pole of $K(z)$, while the function f satisfies the compatibility condition

$$\iint_G \frac{P_l(f)}{r^{2l-n+1}}\, dS < \infty,$$

where P_l is a certain differential operator of order $l - n/2$ with constant coefficients, depending only on the opening of the cone and on the principal part of L, B_j at the point O.

Theorem 1 is proved in the following way. We consider problem (1), (2) in the case when L and B_j are homogeneous operators with constant coefficients, and the domain G is the cone $a_0 x_n^2 = \sum_{i=0}^{n-1} x_i^2$. The functions f, φ_j are assumed—

are equal to zero outside some neighborhood of the origin. Make in (1), (2) the change of variables $t = \ln \frac{1}{r}$. Equations (1), (2) pass into

$$L_0^* u = f e^{-2mt} = F; \tag{1'}$$

$$B_j^* u = \varphi_j e^{-m_j t} = \Phi_j, \tag{2'}$$

where L_0^* and B_j^* are differential operators whose coefficients do not depend on t. After the substitution, the domain G becomes the infinite cylinder $-\infty < t < +\infty$. Making in $(1')$, $(2')$ the Fourier transform with respect to t, we arrive at the problem

$$\widetilde{L}_0(\lambda, D\omega_i) = \widetilde{F}; \tag{1”}$$

$$\widetilde{B}_j(\lambda, D\omega_i) = \widetilde{\Phi}_j, \tag{2”}$$

where $\widetilde{L}_0, \widetilde{B}_j$ are operators of orders $2m$ and m_j, respectively, with coefficients depending on the parameter λ. After all transformations the domain G becomes a sphere. It can be verified that for problem $(1'')$, $(2'')$ the Lopatinskii conditions hold. As is known, problem $(1'')$, $(2'')$ is normally solvable. It can be shown that the inverse operator $K(\lambda)$ is a meromorphic function of λ, bounded in each half-strip $|\operatorname{Im}\lambda| < c$, $|\operatorname{Re}\lambda| > c_1$, where c_1 depends on c. We note that for $n = 2$, $K(\lambda)$ can be written explicitly. Denote by $\widetilde{\psi}$ the vector $[\widetilde{F}, \widetilde{\Phi}_j]$. Put

$$u = \int_{-\infty+ih}^{+\infty+ih} e^{i\lambda t} K(\lambda)\widetilde{\psi}(\lambda, \omega)\, d\lambda, \qquad h = l - \frac{\alpha}{2} - \frac{n}{2}.$$

It is then shown that u is a solution of problem $(1'')$, $(2'')$, satisfying the inequality

$$\|u\|_G^{l,\alpha} \leq \|f\|_G^{l-2m,\alpha} + \|\varphi\|_\Gamma^{l-1/2-m_j,\alpha}.$$

Thus problem (1), (2) is solved in the case of a specially chosen domain. Now one can construct a regularizer by the method of papers ([1,2]), and from its existence normal solvability follows. By the method of decomposition of the identity one obtains estimate (3). For the solution obtained, expansion (4) is established. From expansion (4) it follows that, even for infinitely differentiable right-hand sides (1), (2), the solution need not be infinitely differentiable. The smoothness of a continuous solution cannot exceed the presence in it of l derivatives, where $l = \min|\operatorname{Im}\lambda_j|$; here the minimum is taken over all j such that $\operatorname{Im}\lambda_j > 0$ and $\operatorname{Im}\lambda_j$ is not an integer. It can be shown that $l \to +\infty$ if the aperture of the cone tends to zero. In a number of exceptional cases it may turn out that all numbers $\operatorname{Im}\lambda_j$ are integers, and then, if the compatibility conditions at the point O are fulfilled for the right-hand sides and the coefficients of L, B_j, the solution will be indefinitely differentiable. This will be so, for example, in the Dirichlet and Neumann problems for a second-order equation whose principal part at the origin is the Laplace operator, and the angle has aperture π/p, p

an integer. For the Dirichlet problem for the equation $\Delta\Delta u = 0$ such angles do not exist.

It is known (³) that for the solution of the Dirichlet problem for a second-order equation in the plane the estimate $\|u\|_{W_2^2} \leq \|f\|_{L_2} + \|u\|_{L_2}$ is valid for domains not containing angles greater than π. From expansion (4) it is shown that for $n \geq 4$ this estimate is valid for domains containing cones of arbitrary aperture.

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Note: Figure translations are in progress. See original paper for figures.

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