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Abstract

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NONNEGATIVE EIGENFUNCTIONS OF THE BELTRAMI-LAPLACE OPERATOR ON SYMMETRIC SPACES OF NONPOSITIVE CURVATURE

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1. Let G be a connected semisimple Lie group with finite center, U its maximal compact subgroup, and $\mathcal{E}$ the homogeneous space G/U. In the space $\mathcal{E}$ there exists an invariant Riemannian metric with respect to which $\mathcal{E}$ is a symmetric space of nonpositive curvature. Denote by B the Beltrami-Laplace operator constructed from this metric, and consider the equation

$$Bf - cf = 0, \tag{1}$$

where f is a twice continuously differentiable function on $\mathcal{E}$, and c is a constant.

Our aim is to study the cone C_c of nonnegative solutions of equation (1). For simplicity of exposition we restrict ourselves here to the case where G is the group of real unimodular matrices of order n. However, the results obtained are of a general nature and are valid for any semisimple group. In the case where G is a complex group, results close to ours were obtained in ([1]).

2. If G is the group of real unimodular matrices of order n, then the group of orthogonal matrices may serve as the subgroup U. The Lie algebra of the group G coincides with the algebra $\mathfrak{G}$ of real matrices of order n with trace zero. One may assume that the Riemannian metric in $\mathcal{E}$ is induced by the invariant scalar product $(\mathfrak{g}_1, \mathfrak{g}_2)$ $(\mathfrak{g}_1, \mathfrak{g}_2 \in \mathfrak{G})$, given by the formula $(\mathfrak{g}_1, \mathfrak{g}_2) = \operatorname{sp} \mathfrak{g}_1 \mathfrak{g}_2$. In the n-dimensional vector space consider the hyperplane $\mathfrak{h}$ defined by the equation $\sum_1^n \xi^i = 0$. This hyperplane is naturally identified with the Lie algebra of the group $H \subset G$ of diagonal matrices with positive diagonal elements. For the scalar product introduced, (ξ_1, ξ_2) $(\xi_1, \xi_2 \in \mathfrak{h})$, we have

$$(\xi_1, \xi_2) = \sum_1^n \xi_1^i \xi_2^i.$$

An element $\xi \in \mathfrak{h}$ will be called an element of general position if, for $i \neq j$, $\xi^i \neq \xi^j$. Denote by Λ the domain in $\mathfrak{h}$ defined by the inequalities $\xi^1 \geq \xi^2 \geq \cdots \geq \xi^n$, and let ρ be the vector in $\mathfrak{h}$ determined by the relation

$$(\rho, \xi) = \frac{1}{2} \sum_{i<j} (\xi^i - \xi^j).$$

Theorem 1. *For $c < -(\rho, \rho)$, equation (1) has no nonnegative solutions.*

Along with equation (1), consider the equation

$$\frac{\partial p}{\partial t} = Bp, \tag{2}$$

and let $p_y(t, x) = p(t, x, y)$ be the fundamental solution of this equation, i.e. the function satisfying, as a function of x, equation (2) for $t > 0$ and the initial condition $p(0, x, y) = \delta_y(x)$, where $\delta_y(x)$ is the δ-function in $\mathcal{E}$ with respect to the invariant volume, concentrated at the point y.

Theorem 2. If $c \geq -(\rho, \rho)$ and $x \neq y$, then

$$\int_0^\infty e^{-ct} p(t, x, y)\, dt$$

converges.

Theorem 2 ensures the existence (for $c \geq -(\rho, \rho)$) of the Green' s function of equation (1).

3. Let Z denote the group of upper triangular matrices whose diagonal elements are equal to 1. As is known, every matrix $g \in G$ is represented, and in a unique way, in the form $g = zhu$, where $z \in Z$, $h \in H$, $u \in U$. Therefore every point $x \in \mathcal{E}$ is represented in the form $x = zhx_0$; here x_0 is the point of the space $\mathcal{E}$ whose stationary subgroup coincides with the subgroup U. (By gx $(g \in G,\ x \in \mathcal{E})$ we denote the point obtained from the point x by the motion g.) For any vector $\lambda \in \mathfrak{h}$ define on the group H the function h^λ by the formula $h^\lambda = \exp(\lambda, \mathfrak{h})$, where $\mathfrak{h} = \ln h$ is the vector whose coordinates are equal to the logarithms of the diagonal elements of the matrix h. Introduce on the space $\mathcal{E}$ the function $\varphi(x, \lambda)$ $(x \in \mathcal{E},\ \lambda \in \mathfrak{h})$, setting $\varphi(x, \lambda) = h^{\lambda+\rho}$ for $x = zhx_0$.

A nonnegative solution $f(x)$ of equation (1) is called **minimal** if every nonnegative solution of this equation not exceeding $f(x)$ can differ from $f(x)$ only by a constant factor.

Theorem 3. Every minimal solution $f(x)$ of equation (1) (for $c \geq -(\rho, \rho)$) is proportional to one of the functions $\varphi(ux, \lambda)$, where $u \in U,\ \lambda \in \Lambda,\ (\lambda, \lambda) = c + (\rho, \rho)$.

From Theorem 3 it follows that

Theorem 4. Every nonnegative solution $f(x)$ of equation (1) can be represented in the form

$$f(x) = \int_{U \times \Lambda_c} \varphi(ux, \lambda)\, \mu(du\, d\lambda),$$

where Λ_c is the intersection of the domain Λ with the sphere $(\lambda, \lambda) = c + (\rho, \rho)$, and $\mu(du\, d\lambda)$ is a finite Borel measure on $U \times \Lambda_c$.

4. A twice continuously differentiable function $f(x)$ on $\mathcal{E}$ will be called **harmonic** if it satisfies the equation $Bf(x) = 0$.

Let $K \subset G$ denote the group of all upper triangular matrices and let Ξ denote the homogeneous space G/K. Since every matrix $g \in G$ is represented in the form $g = uk$ $(u \in U,\ k \in K)$, the group U acts transitively on Ξ. Therefore the stationary subgroup U_x of any point $x \in \mathcal{E}$ acts transitively on Ξ. By compactness of U_x, on Ξ there exists, and is unique, a measure invariant with respect to U_x and normalized so that the measure of the whole space Ξ is equal to 1. This measure depends on the point $x \in \mathcal{E}$, and we denote it by $\mu(x, d\xi)$.

Theorem 5. Every bounded harmonic function $f(x)$ on $\mathcal{E}$ can be represented in the form

$$f(x) = \int_{\Xi} p(\xi)\, \mu(x, d\xi), \tag{3}$$

where $p(\xi)$ is an arbitrary function on Ξ integrable with respect to the measures $\mu(x, d\xi)$. The function $p(\xi)$ is determined by the function $f(x)$ uniquely almost everywhere on Ξ*.

From Theorem 5 it follows easily that

Theorem 6. Every bounded harmonic function is annihilated by all Laplace operators on the space $\mathcal{E}$ by which the function $f(x) \equiv 1$ is annihilated.

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* Cf. Theorem 10 of (2).

Theorem 6 was obtained by other methods in ([3]).

Let $\mathfrak{h}$ be an arbitrary element in general position belonging to Λ. Denote by $h(t)$ the corresponding one-parameter subgroup in the group H.

Theorem 7. *If the function $f(x)$ is given by formula (3) with a continuous function $p(\xi)$, then*

$$\lim_{t\to+\infty} f(h(t)x_0) = p(\xi_0),$$

where ξ_0 is the point Ξ whose stationary subgroup coincides with the group K.

Theorem 7 makes it possible to solve the question of the behavior of the function $f(x)$ as $x \to \infty$ along a geodesic γ in general position in the space $\mathcal{E}$. It turns out that the values of the function $f(x)$ then tend to a certain value $p(\xi)$. (The point ξ depends on the geodesic γ.) It is therefore natural to call the values $p(\xi)$ of the function p the **boundary values** of the function $f(x)$.

Let now γ be an arbitrary directed geodesic of the space $\mathcal{E}$, and let F be a finite pencil of geodesics containing γ (see ([2])). According to ([2]), the aggregate of null pencils $\Gamma \subset F$ forms a certain Riemannian symmetric space $\mathcal{F}$ of nonpositive curvature. Like every such space, $\mathcal{F}$ decomposes into the direct product $E \times \mathcal{E}'$ of a Euclidean space E and a Riemannian symmetric space $\mathcal{E}'$ of nonpositive curvature with a semisimple group of motions. Denote by

$$\rho : \mathcal{F} \to \mathcal{E}'$$

the projection of the space $\mathcal{F}$ onto the space $\mathcal{E}'$. To each geodesic $\gamma \in F$ there corresponds a point $x' = \pi(\gamma) \in \mathcal{E}'$ by the rule $\pi(\gamma) = \rho(\Gamma)$, where Γ is the null pencil containing the geodesic γ. We shall write $x \to +\infty$, $x \in \gamma$, if x tends to infinity along the directed geodesic γ in the positive direction.

Theorem 8. *If $f(x)$ is a bounded harmonic function with continuous boundary values and if, for two geodesics γ_1 and γ_2 from a finite pencil F, $\pi(\gamma_1) = \pi(\gamma_2)$, then*

$$\lim_{x\to+\infty,\ x\in\gamma_1} f(x) = \lim_{x\to+\infty,\ x\in\gamma_2} f(x).$$

Theorem 8 shows that the limiting values of the function $f(x)$ induce a certain function $f'(x')$ on the space $\mathcal{E}'$.

Theorem 9. *If $f(x)$ is a bounded harmonic function with continuous boundary values, then its limiting values $f'(x')$ on the space $\mathcal{E}'$ form a function harmonic on $\mathcal{E}'$.*

Theorems 7-9 are immediate consequences of Theorems 11, 9, and 12 from ([2]). We note that the basis for obtaining the results set forth is Martin' s method ([4]) (see also ([5])) and the method of horispherical radial parts of Laplace operators on symmetric spaces ([6]).

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Note: Figure translations are in progress. See original paper for figures.