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Abstract

Full Text

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SOME QUESTIONS ON THE OSCILLATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. We shall consider the equation

$$Lx \equiv x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x \equiv 0 \tag{1}$$

with continuous coefficients $p_1(t), \dots, p_n(t)$. In what follows, when speaking of solutions of equation (1), we shall have in mind only nontrivial solutions. Wherever the number of zeros of any functions is discussed, it is assumed that each zero is counted according to its multiplicity. By $M(c, k; d, l)$ we denote the class of functions positive for $c < t < d$ and having at the point c a zero of multiplicity not less than k, and at the point d a zero of multiplicity not less than l. The collection of functions n-times continuously differentiable on some interval I is denoted, as usual, by $C^n(I)$. We shall also use the notation

$$q_+(t) = \max\{0, q(t)\}, \qquad q_-(t) = \max\{0, -q(t)\}.$$

2. As is known, an interval $a \le t \le b$ is called a nonoscillation interval for the operator L if every solution of (1) has on $[a, b]$ no more than $n-1$ zeros; otherwise $[a, b]$ is called an oscillation interval for L.

The question of nonoscillation is directly connected with the multipoint Vallée-Poussin boundary value problem, which consists in finding a solution of the equation $Lx = f$ taking prescribed values at n prescribed points, and with the generalized Vallée-Poussin boundary value problem:

$$Lx = f(t) \qquad (a \le t \le b), \tag{2}$$

$$x(a_i) = A_{i,0}, \quad x'(a_i) = A_{i,1}, \dots, x^{(r_i-1)}(a_i) = A_{i,r_i-1}, \quad i = 1, 2, \dots, m \tag{3}$$

$$(a \le a_1 < a_2 < \dots < a_m \le b; \quad r_1 + r_2 + \dots + r_m = n).$$

If $[a, b]$ is a nonoscillation interval for L, then problem (2)–(3), obviously, is solvable for arbitrary right-hand sides, and in a unique manner. It is known that, in the absence of nontrivial solutions of (1) satisfying the corresponding homogeneous conditions

$$x(a_i) = x'(a_i) = \cdots = x^{(r_i-1)}(a_i) = 0, \quad i = 1, 2, \dots, m, \tag{4}$$

there exists a Green' s function $G(t, s)$, with the aid of which the solution of problem (2)–(4) is written in the form

$$x(t) = \int_a^b G(t, s) f(s)\, ds.$$

Essential for comparison theorems and for various estimates is the following fact: if and only if $[a, b]$ is a nonoscillation interval for L, then for any $a_1, \dots, a_m$ from $[a, b]$ the Green' s function $G(t, s)$ is of constant sign as a function of s in the square $a \le t, s \le b$; the sign of $G(t, s)$ is determined by the relation

$$G(t, s)(t - a_1)^{r_1}(t - a_2)^{r_2} \cdots (t - a_m)^{r_m} \ge 0.$$

This result was obtained by the author (reported at the IV All-Union Mathematical Congress in a joint report by Perov, Kibenko, Krasnosel' skii, and Levin) and independently by E. S. Chichkin (see ([7])). For $n = 3$ analo-

···logical proposition was established earlier by N. V. Azbelev. From the theorem on the sign-constancy of the Green' s function there follows the following proposition.

Lemma 1. *In order that $[a, b]$ be an interval of oscillation for L, it is necessary and sufficient that there exist a function $y(t) \in C^n[a, b]$ satisfying the following conditions:*

1) *$y(t)$ has on $[a, b]$ at least n zeros;*

$$y(a_i) = \dots = y^{(r_i-1)}(a_i) = 0, \quad i = 1, 2, \dots, m \quad (a \le a_1 < \dots < a_m \le b;$$

$$r_1 + r_2 + \dots + r_m = n);$$

2) *$y(t)$ is not identically equal to zero on $[a_1, a_m]$, $y^{(r_m)}(a_m) \le 0$; $Ly \ge 0$ on $[a_1, a_m]$.*

3. For any point a which is the left endpoint of some interval of oscillation, one can define the conjugate point $\bar{a}$ as the infimum of those c $(c > a)$ for which $[a, c]$ is an interval of oscillation for L. Clearly, $\bar{a} > a$ and $[a, \bar{a}]$ is an interval of oscillation.

Theorem 1. *If the point a has a conjugate point, then there exists a solution of equation (1) belonging to the class* $M(a, k; \bar{a}, n-k)$ *for some* k $(1 \leq k \leq n-1)$.

For applications of this theorem, the positivity of the solution in question inside $(a, \bar{a})$ is essential. Theorem 1 gives rise naturally to the following definition.

We shall say that $[a, b]$ is an interval of $(k, n-k)$-oscillation for L (or that L has on $[a, b]$ oscillation of type $(k, n-k)$) if $b \geq \bar{a}$ and there exists a solution of (1) belonging to the class $M(a, k; \bar{a}, n-k)$. We shall call the type of oscillation even or odd according as the number $n-k$ is even or odd. In accordance with Theorem 1, for every interval of oscillation there is thus determined one or several of the $n-1$ possible types of oscillation $(n-1, 1)$, $(n-2, 2), \dots, (1, n-1)$. Every interval of $(k, n-k)$-oscillation for L is, as is easy to see, an interval of $(n-k, k)$-oscillation for the adjoint operator

$$\bar{L}x = (-1)^n x^{(n)} + (-1)^{n-1}(p_1 x)^{(n-1)} + \dots$$

$$\dots - (p_{n-1}x)' + p_{nx}.$$

In particular, for a self-adjoint operator every interval of $(k, n-k)$-oscillation is simultaneously an interval of $(n-k, k)$-oscillation.

4. **Lemma 2.** *Let* $[a, \bar{a}]$ *be an interval of* $(k, n-k)$*-oscillation for* L. *Then for every* $x(t) \in C^n[a, \bar{a}]$ *having a zero of multiplicity not less than* k *at the point* a *and a zero of multiplicity not less than* $n-k$ *at the point* $\bar{a}$, *and not being a solution of equation (1), the function* $h(t) = Ly$ *changes sign on* $[a, \bar{a}]$.

Theorem 2. *Let the operators* L_1 *and* L_2 *be defined by the formulas*

$$L_i x \equiv x^{(n)} + q_1(t)x^{(n-1)} + \dots + q_{n-1}(t)x' + q_{n,i}(t)x, \tag{5}$$

$i = 1, 2$, *where* $q_{n,1}(t) \leq q_{n,2}(t)$.

Then:

a) *every interval of odd oscillation for* L_1 *is an interval of odd oscillation for* L_2;

b) *every interval of even oscillation for* L_2 *is an interval of even oscillation for* L_1.

The proof of Theorem 2 is based on the use of Lemmas 1, 2 and Theorem 1. For $n = 2$ assertion a) becomes Sturm' s comparison theorem.

Corollary 1. *Let the operators* L_1, L_2, L_3 *be defined by the formulas (5),* $i = 1, 2, 3$, *with* $q_{n,1}(t) \leq q_{n,2}(t) \leq q_{n,3}(t)$. *Then every interval that is an interval* *of nonoscillation for each of the operators L_1, L_3 is an interval of nonoscillation for L_2.*

Indeed, the assumption that L_2 has an odd oscillation on this interval leads to a contradiction with assertion a) of Theorem 2 as applied to L_2, L_3, while the assumption of an even oscillation of L_2 leads to a contradiction with assertion b) as applied to L_1, L_2. For operators of the form $x^{(n)} + q(t)x$, a statement analogous to Corollary 1 was obtained by V. A. Kondrat' ev ([6]); Kondrat' ev' s method of proof is apparently applicable also in the general case.

Corollary 2. *For $q(t) \geq 0$, the operator $x^{(n)} + q(t)x$ can have only odd oscillation, and for $q(t) \leq 0$, only even oscillation.*

For the proof it suffices to assume the contrary and compare the operators $x^{(n)} + q(t)x$, $x^{(n)}$.

5. Corollary 2, together with the use of self-adjointness (for even n), makes it possible to reduce substantially the number of possible types of oscillation for operators of the form $x^{(n)} + q(t)x$ with sign-constant $q(t)$. For small n, in several cases one can completely determine the type of oscillation (if, of course, it occurs), relying only on the sign of $q(t)$. For $n > 2$ there are five such cases of definiteness of the oscillation type: 1) $n = 3$, $q(t) \geq 0$—type $(2, 1)$; 2) $n = 3$, $q(t) \leq 0$—type $(1, 2)$; 3) $n = 4$, $q(t) \geq 0$—type $(3, 1)$ and simultaneously $(1, 3)$; 4) $n = 4$, $q(t) \leq 0$—type $(2, 2)$; 5) $n = 6$, $q(t) \leq 0$—type $(4, 2)$ and simultaneously $(2, 4)$.

6. **Lemma 3.** *If some function $y(t)$, positive for $a < t \leq b$, has at the point a a zero of multiplicity not less than $n-2$, and $Ly \leq 0$ on $[a, b]$, then $[a, b]$ is not an interval of $(n-1, 1)$-oscillation for L.*

For $b = \bar{a}$, the existence of such a $y(t)$ is not only sufficient but also necessary for the assertion of the lemma. The use of the adjoint operator $\bar{L}$ makes it possible, analogously, to exclude oscillation of type $(1, n-1)$ for L. Hence there immediately follow necessary and sufficient, though non-effective, criteria for non-oscillation for an operator of third order and for the operator $x^{\mathrm{IV}} + q(t)x$ for $q(t) \geq 0$, i.e., for the cases in which $(n-1, 1)$ and $(1, n-1)$ are the only possible types of oscillation (see ([2–5])). An effective sufficient condition for non-oscillation in the general case was indicated by Vallée-Poussin ([1]); two different strengthenings of the Vallée-Poussin theorem are given in ([8],[9]).

7. Analogously to the notion introduced above of the point $\bar{a}$ conjugate to a, lying to the right of a, one can introduce the notion of a point $\underline{a}$, conjugate to a from the left, defined as the supremum of those c ($c < a$) for which $[c, a]$ is an interval of oscillation for L. Everything stated above carries over, with the corresponding changes, to points conjugate from the left; in particular, for any interval of oscillation $[a, b]$ one can define the type of oscillation from the left as the type of oscillation of L on $[\underline{b}, b]$, etc. It can be shown that $\bar{a}$, as a function of a, is strictly increasing, whence it follows immediately that $\underline{\bar{a}} = a$.

Let us note that, for third-order equations, a number of results analogous to those given above were obtained by N. V. Azbelev and Z. B. Tsaljuk ([5]), who, in particular, used for $n = 3$ a notion analogous to the type of oscillation.

8. As is known, Chaplygin' s theorem on differential inequalities is valid for the operator L on $[a, b]$ if and only if the Cauchy function is nonnegative, i.e., if every solution of (1) having at some point t_0 of $[a, b]$ a zero of multiplicity $n - 1$ does not change sign on $[t_0, b]$. We give one effective condition for the validity of Chaplygin' s theorem for operators of the form $x^{(n)} + q(t)x$.

Theorem 3. *If the inequality*

$$\int_a^b q_+(t)\,dt \leq \frac{4^{n-1}(n-1)!}{(b-a)^{n-1}},$$

then for the operator $Lx \equiv x^{(n)} + q(t)x$ on $[a, b]$ Chaplygin' s theorem is valid.

The theorem formulated above can be used to obtain certain effective criteria for non-oscillation.

Corollary 1. If the value of each of the integrals $\int_a^b q_+(t)\,dt$, $\int_a^b q_-(t)\,dt$ does not exceed $32(b-a)^{-2}$, then $[a, b]$ is an interval of non-oscillation for the operator $x''' + q(t)x$.

Corollary 2. If $q(t) \geqslant 0$ and $\int_a^b q(t)\,dt \leqslant 384(b-a)^{-3}$, then $[a, b]$ is an interval of non-oscillation for the operator $x^{\mathrm{IV}} + q(t)x$.

The values of the constants in Theorem 3 and its corollaries cannot be improved, as is shown by the case

$$q(t) = 4^{n-1}(n-1)!(b-a)^{1-n}\delta\left(t - \frac{a+b}{2}\right),$$

where $\delta(t)$ is the delta function.

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Note: Figure translations are in progress. See original paper for figures.