Z. I. Rekhlickii

In our notes ([1]-), criteria were obtained for the stability of solutions of differential equations of the form

1963

SovietRxiv

Abstract

Full Text

Z. I. Rekhlickii

SPECTRAL CRITERIA FOR THE STABILITY OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

(Presented by Academician I. G. Petrovskii on 3 X 1962)

In our notes ([1]-[4]), criteria were obtained for the stability of solutions of differential equations of the form

$$\frac{dy}{dt} - \int_0^\infty y(t-s)\,dr(t,s) = f(t) \qquad (0 \le t < \infty),$$

$$y^{(n)} - \sum_{k=0}^{n-1} p_k(t) y^{(k)}(t - a_k) = f(t) \qquad (0 \le t < \infty)$$

in the case when $p_k'(t) \to 0$ and $\partial r(t,s)/\partial t \to 0$ as $t \to \infty$. Therefore these results cannot be applied to differential equations with periodic coefficients.

In the present note we give necessary and sufficient criteria for boundedness of solutions on the half-axis $0 \le t < \infty$ of differential equations of the form

$$\frac{dy}{dt} - A(t)y = f(t), \qquad y^{(n)} - \sum_{k=0}^{n-1} p_k(t) y^{(k)}(t) = f(t),$$

where $A(t+a) = A(t)$; $p_k(t+a) = p_k(t)$ $(a > 0)$.

Theorem 1. Consider the boundary-value problem:

$$\frac{dy}{dt} - A(t)y(t) = f(t) \qquad (0 \le t < \infty), \tag{1}$$

$$y(0) = y_0.$$

Let $A(t)$ be a continuous operator-function acting in a complex Banach space $\tilde{\mathcal{E}}$, satisfying the condition $A(t+a) = A(t)$ $(a > 0)$; let $f(t)$, $y(t)$ be continuous functions with values belonging to $\tilde{\mathcal{E}}$.

In order that the boundary-value problem (1) have a bounded solution $y(t)$ for all bounded $f(t)$, it is necessary and sufficient that all points of the spectrum $\lambda(\theta)$ of the family of operators

$$\left\{\exp\left[\int_0^a A(s+\theta)\,ds\right]\right\}$$

for $0 \le \theta \le a$ lie inside the unit circle,

$$|\lambda(\theta)| < 1 \qquad (0 \le \theta \le a), \qquad \lambda(\theta) \in \operatorname{sp}\exp\left[\int_0^a A(s+\theta)\,ds\right], \tag{2}$$

where

$$\exp\left[\int_0^a A(s+\theta)\,ds\right]$$

is the multiplicative integral.

Proof. Consider the auxiliary function

$$\Phi(t,z) = \sum_{k=0}^{\infty} y(t+\theta+ka)z^k,$$

which, being a solution of the equation

$$\frac{\partial\Phi}{\partial t} - A(t+\theta)\Phi = \sum_{k=0}^{\infty} f(t+\theta+ka)z^k,$$

can be represented in the form

$$\Phi(t,z) = \int_0^t \exp\left[\int_\tau^t A(s+\theta)\,ds\right]\sum_{k=0}^{\infty} f(\tau+\theta+ka)z^k\,d\tau + \exp\left[\int_0^t A(s+\theta)\,ds\right]\Phi(0,z) \quad (0 \le \theta \le a).$$

For $t = a$ we shall have

$$\Phi(a,z) = \int_0^a \exp\left[\int_\tau^a A(s+\theta)\,ds\right]\sum_{k=0}^{\infty} f(\tau+\theta+ka)z^k\,d\tau + \exp\left[\int_0^a A(s+\theta)\,ds\right]\Phi(0,z).$$

But

$$\Phi(0,z) = \sum_{k=0}^{\infty} y(\theta+ka)z^k = y(\theta) + z\sum_{k=0}^{\infty} y(\theta+(k+1)a)z^k = y(\theta) + z\Phi(a,z).$$

Therefore, for the auxiliary function

$$\Phi(a, z) = \sum_{k=0}^{\infty} y(\theta + (k+1)a) z^k$$

the formula holds

$$\sum_{k=0}^{\infty} y(\theta + (k+1)a) z^k = \left(I - z \exp\left[\int_0^a A(s+\theta)\, ds \right] \right)^{-1} F(\theta, z), \qquad (3)$$

where

$$F(\theta, z) = \int_0^a \exp\left[\int_\tau^a A(s+\theta)\, ds \right] \sum_{k=0}^{\infty} f(\tau + \theta + ka) z^k \, d\tau +$$

$$+ \exp\left[\int_0^a A(s+\theta)\, ds \right] \left(\int_0^\theta \exp\left[\int_\tau^\theta A(s+\theta)\, ds \right] f(\tau)\, d\tau + \exp\left[\int_0^\theta A(s+\theta)\, ds \right] y_0 \right).$$

It is easy to see that the function

$$F(\theta, z) + \sum_{n=0}^{\infty} f_k(\theta) z^k$$

is holomorphic for $|z| < 1$ and has uniformly bounded coefficients $\|f_k(\theta)\| \leq C$, if the function $f(t)$ is bounded:

$$\|f(t)\| \leq C_1, \qquad (0 \leq t < \infty).$$

Consider the operator-function

$$B(\theta, z) = \left(I - z \exp\left[\int_0^a A(s+\theta)\, ds \right] \right)^{-1} = \sum_{k=0}^{\infty} B_k(\theta) z^k.$$

Suppose condition (2) is satisfied; then the operator-function $B(\theta, z)$ will be holomorphic for $|z| \leq \rho < 1$. On the basis of Cauchy' s inequality we shall have

$$\|B_k(\theta)\| \leq \frac{M}{\rho^k} = M q^k,$$

where $M = \max \|B(\theta, z)\|$, $0 < q = \frac{1}{\rho} < 1$, $|z| = \rho$, $0 \leq \theta \leq a$.

Applying formula (3), one can estimate $y(\theta+(k+1)a)$ in terms of $\|f_k(\theta)\|$ and $\|B_k(\theta)\|$:

$$\|y(\theta+(k+1)a)\| \le \left(\sum_{k=0}^{\infty}\|B_k(\theta)\|\right)\|f_k(\theta)\| \le \frac{MC}{1-q} = C_2 < \infty,$$

i.e. $y(t)$ is bounded for $0 \le t < \infty$. In this case Theorem 1 is proved.

Suppose that for some $\theta=\theta_1$

$$|\lambda(\theta_1)| \ge 1, \qquad \lambda(\theta_1) \in \operatorname{sp}\exp\left[\int_0^a A(s+\theta_1)\,ds\right].$$

We shall assume that

$$\max_{0\le\theta\le a}|\lambda(\theta)| = |\lambda(\theta_1)| \ge 1.$$

Then the operator-function $B(\theta_1,z)$ has a singularity at $z=z_0=1/\lambda(\theta_1)$, and is holomorphic for all z with $|z|<|z_0|$.

Consider $z_n \to z_0$ ($|z_n|<|z_0|$) as $n\to\infty$. It can be shown that $\|B(\theta_1,z_n)\|\to\infty$ as $n\to\infty$; therefore

$$\|B(\theta_1,z_n)\vec{e}_n\| \longrightarrow_{n\to\infty} \infty, \qquad \|\vec{e}_n\|=1.$$

Using formula (3), construct a sequence $\|f_n(t)\| \le C$ for which the solutions $y_n(t)$ of equation (1) will be unbounded:

$$\overline{\lim}\,\|y_n(t)\| = \infty \quad \text{as } n,t\to\infty.$$

Consider two cases:

1) Suppose $|z_0|<1$ ($|\lambda(\theta_1)|>1$); then, for $z=z_n$, $\theta=\theta_1$, and

$$f_n(\tau+\theta_1) = \exp\left[\int_a^\tau A(s+\theta_1)\,ds\right]\vec{e}_n\varphi(\tau),$$

where $\varphi(\tau)=0$ for $\tau\le 0,\ \tau\ge a$, formula (3) takes the form

$$\sum_{k=0}^{\infty} y_n(\theta_1+(k+1)a)z_n^k = B(\theta_1,z_n)\vec{e}_n\int_0^a \varphi(\tau)\,d\tau, \qquad \int_0^a\varphi(\tau)\,d\tau>0. \quad (4)$$

2) Suppose $|z_0|=1$ ($|\lambda(\theta_1)|=1$); then, for $z=z_n$, $\theta=\theta_1$, and

$$f_n(\tau+\theta_1+ka)=\exp\left[\int_a^\tau A(s+\theta_1)\,ds\right]\vec{e}_n z_0^{-k}\varphi(\tau),\quad \varphi(\tau)=0\ (\tau\le 0),$$

formula (3) takes the form

$$\sum_{k=0}^{\infty} y_n(\theta_1+(k+1)a)z_n^k=B(\theta_1,z_n)\vec{e}_n\frac{\displaystyle\int_0^a\varphi(\tau)\,d\tau}{1-z_0^{-1}z_n},\qquad \int_0^a\varphi(\tau)\,d\tau>0. \quad (5)$$

From (4) and (5) it follows that

$$\overline{\lim}_{n,t\to\infty}\|y_n(t)\|=\infty.$$

Now one can show that the boundary-value problem (1), for some bounded function $f(t)$, has an unbounded solution $y(t)$. Indeed, if the boundary-value problem, for all bounded $f(t)$, had a bounded solution $y(t)$, then the operator determining the solution $x(t)$ would be bounded in norm, and from $\|f_n(t)\|\le C$ it would follow that $\|y_n(t)\|\le C_1$, but this is impossible, since we have shown that

$$\lim_{n,\ t\to\infty}\|y_n(t)\|=\infty.$$

Theorem 1 is proved.

Corollary. If in Theorem 1 $\bar{A}(t_1)\bar{A}(t_2)=\bar{A}(t_1)\bar{A}(t_1)$, then condition (2) may be replaced by the simpler one: all points $\lambda\in\operatorname{sp}\displaystyle\int_0^a\bar{A}(s)\,ds$ must satisfy the condition $\operatorname{Re}\lambda<0$.

Theorem 2. *Consider the boundary-value problem*

$$y^{(n)}-\sum_{k=0}^{n-1}p_k(t)y^{(k)}(t)=f(t)\qquad(0\le t<\infty), \quad (6)$$

$$y^{(k)}(0)=y_k\qquad(k=0,1,\dots,n-1),$$

where $p_k(t+a)=p_k(t)$ are continuous complex functions ($a>0$); $f(t)$ is a continuous scalar function.

In order that the boundary-value problem (6) have a bounded solution for all bounded $f(t)$, it is necessary and sufficient that the coefficients $p_k(t)$ satisfy the condition

$$|\lambda(\theta)| < 1 \quad \text{for } 0 \le \theta < a, \qquad \lambda(\theta) \in \operatorname{sp}\exp\left[\int_0^a A(s+\theta)\,ds\right],$$

where $A(t) = \|p_{ik}(t)\|_1^n$ *is the coefficient matrix, which has the form*

$$p_{1k}(t) = p_{n-k}(t), \qquad p_{i,i-1}(t) = 1 \quad (i > 1), \qquad p_{ik}(t) = 0 \quad (i > 1,\ k \ne i-1).$$

Proof. Consider the column vectors

$$u = (y^{(n-1)}, y^{(n-2)}, \dots, y', y), \qquad \varphi(t) = (f(t), 0, \dots, 0, 0).$$

Then the boundary-value problem (6) can be written in the form

$$\frac{du}{dt} - A(t)u = \varphi(t) \qquad (0 \le t < \infty), \tag{1'}$$

$$u(0) = u_0,$$

where $A(t) = \|p_{ik}(t)\|_1^n$ is the coefficient matrix.

In view of the fact that boundedness of the solution $y(t)$ of the boundary-value problem (6) implies boundedness of all derivatives $y^{(k)}(t)$ (see (5)), the boundary-value problem (6), with respect to stability of solutions, is equivalent to problem $(1')$, to which Theorem 1 is applicable.

Odessa Hydrometeorological Institute

Received
28 IX 1962

REFERENCES

1. Z. I. Rekhlitskii, DAN, **111**, No. 1 (1956).
2. Z. I. Rekhlitskii, DAN, **118**, No. 3 (1958).
3. I. M. Rekhlitskii, DAN, **125**, No. 1 (1959).
4. Z. I. Rekhlitskii, DAN, **127**, No. 5 (1959).
5. M. A. Rutman, *Collection of articles on contemporary problems of constructive function theory*, Moscow, 1961.