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Abstract

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APPLICATION OF A COMPLEX VARIABLE TO THE STUDY OF THE BEHRENS-FISHER PROBLEM

The Behrens-Fisher problem is one of the best-known problems in the study of statistical similar tests used for eliminating nuisance parameters. On problems of this type, see the report of J. Neyman (^{1}), where an extensive bibliography is also indicated.

The construction of statistical tests and confidence regions for parametric problems in most cases is connected with repeated sampling and with the existence of sufficient statistics of finite rank. In this case one can choose sufficient statistics whose distribution will be an analytic function of the parameters.

For the study of similar tests based on these sufficient statistics, one may propose the investigation of analytic continuations of integral relations expressing the similarity of tests with respect to the parameters. This makes it possible in many cases to study rather deeply the nature of such tests.

In the present note an example of such an investigation is given in application to the Behrens-Fisher problem.

Let $x_1, \dots, x_{n_1} \in N(a_1, \sigma_1)$, $y_1, \dots, y_{n_2} \in N(a_2, \sigma_2)$ be two independent normal repeated samples. As is known, the general Behrens-Fisher problem consists in studying tests for the hypothesis $H_0 : a_1 = a_2$, similar with respect to σ_1 and σ_2. Here, in particular, the Bartlett-Romanovsky-Scheffé test (^{2}) is known. Of special interest is the case of nontrivial similar tests depending only on four sufficient statistics:

$$\bar{x} = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i; \qquad \bar{y} = \frac{1}{n_2} \sum_{j=1}^{n_2} y_j;$$

$$s_1^2 = \frac{1}{n_1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2; \qquad s_2^2 = \frac{1}{n_2} \sum_{j=1}^{n_2} (y_j - \bar{y})^2.$$

In particular, the article by A. Wald (^{3}) is devoted to this. In it, similar statistics of a special kind are considered, depending only on the ratios $\dfrac{\bar{x} - \bar{y}}{s_2}$ and $\dfrac{s_1}{s_2}$, i.e. akin to the well-known Student ratio

$$\frac{\bar{x}-\bar{y}}{\sqrt{n_1 s_1^2 + n_2 s_2^2}}.$$

In the present note we shall give an example of the investigation of such similar statistics by means of the method of analytic continuation with respect to a parameter.

We shall consider similar statistics $u = u(\bar{x}-\bar{y}, s_1^2, s_2^2)$ under the following conditions:

I. For any $\xi > 0$ we have:

$$u((\bar{x}-\bar{y})\sqrt{\xi},\ s_1^2\xi,\ s_2^2\xi) \equiv u(\bar{x}-\bar{y},\ s_1^2,\ s_2^2)$$

(weighted homogeneity). Hence it follows that $u = u_1(\xi_1, \xi_2)$, where

$$\xi_1 = \frac{\bar{x}-\bar{y}}{s_2}; \qquad \xi_2 = \frac{s_1^2}{s_2^2}.$$

II. The function $u_1(\xi_1, \xi_2)$ is discontinuous in the half-plane Ω': $-\infty < \xi < \infty$; $0 \leqslant \xi_2 < \infty$.

Here it should be noted that nontrivial similar tests for the Behrens-Fisher problem that would be discontinuous in $x_1, \dots, x_{n_1}$; $y_1, \dots, y_{n_2}$ do not exist. This is not hard to derive from the Lehmann-Scheffé theorem on complete systems of sufficient statistics. Thus our function $u_1(\xi_1, \xi_2)$, discontinuous in ξ_1, ξ_2, will not be a discontinuous function of the original sample points.

III. The function $u_1(\xi_1, \xi_2)$ assumes arbitrarily large values in the half-plane Ω'.

In view of the fact that any discontinuous function of a similar statistic is itself a similar statistic, this condition may be replaced by another.

III′. The function $u_1(\xi_1, \xi_2)$ does not attain its maximum in the half-plane Ω' at a finite distance from the origin.*

Apparently, similar statistics $u = u(\bar{x}-\bar{y}, s_1^2, s_2^2)$ of the type described (in particular, those studied by A. Wald) do not exist. However, the method set forth does not yet make it possible to detect this. We shall introduce more stringent conditions, partly following A. Wald (^{3}), who considered analytic statistics.

For what follows, let us change notation: $\xi_1 = \xi$; $\eta = \sqrt{\xi_2} = s_1/s_2$. We shall denote the half-plane $-\infty < \xi < \infty$, $0 \leqslant \eta < \infty$ by Ω, and the open half-plane $-\infty < \xi < \infty$, $0 < \eta < \infty$ by Ω^0.

Theorem. Let the statistic $g(\xi, \eta) = u_1(\xi_1, \xi_2)$, besides satisfying requirements I, II, III, also satisfy the following:

IV. $g(\xi, \eta)$ is three times continuously differentiable in Ω^0, with the possible exception of a finite number of points, and is continuous in Ω.

V. The number of singular points, i.e., points satisfying the condition $\partial g/\partial\xi = \partial g/\partial\eta = 0$, is finite.

VI. In the vertical strip $|\xi| \leqslant 4$, the level lines $g(\xi,\eta) = C$ for sufficiently large C constitute a number of analytic curves bounded independently of C.

VII. For $|\xi| \leqslant 4,\ |\eta| > C$,

$$|\operatorname{grad} g| > \eta^{-3/2(n_2+1)}$$

and

$$\int\limits_{\substack{L \\ \eta>C}} \frac{dL}{\eta^{2N-n_1+2}|\operatorname{grad} g|} < \infty,$$

where L is any of the arcs $g(\xi,\eta) = C$.

Under these conditions the statistic $g(\xi,\eta)$ cannot be similar for the Behrens-Fisher problem.

Let us briefly outline the proof of this theorem. Let n_1, n_2 be the sizes of such samples; let $n_2 \geqslant n_1 \geqslant 2$;

$$\sigma_1^2 = \theta_1; \qquad \sigma_2^2 = \theta_2; \qquad X_1 = (\bar{x} - \bar{y})\left(\frac{n_1\theta_2 + n_2\theta_1}{n_1 n_2}\right)^{-1/2};$$

$$U = \left(\frac{n_1}{\theta_1}\right)^{1/2} s_1; \qquad V = \left(\frac{n_2}{\theta_2}\right)^{1/2} s_2; \qquad \xi = \frac{X_1}{V}; \qquad \eta = \frac{U}{V}.$$

Then the joint probability density $\rho(\xi,\eta)$ of the random variables ξ and η is given by the formula

$$\rho(\xi,\eta) = C_{n_1,n_2} \frac{\eta^{n_1-2}}{(1+\xi^2+\eta^2)^{(n_1+n_2)/2-1/2}}$$

(C_{n_1,n_2} is a normalizing constant).

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* This remark was communicated to the author by I. V. Romanovskii.

Let a constant $C > 1$ be given and let $\Delta C > 0$ (a small increment); let $\psi(g) = 1$ for $C \leqslant g \leqslant C + \Delta C$ and $\psi(g) = 0$ in all other cases. Further, set $\dfrac{n_2}{n_1}\dfrac{\theta_2}{\theta_1} = \theta > 0$. The similarity conditions for the statistic $g(\xi,\eta)$ will consist in the fact that

$$\iint_\Omega \psi\left(g\left(\xi\sqrt{1+\theta},\ \eta\sqrt{\theta}\right)\right)\rho(\xi,\eta)\,d\xi\,d\eta = C_\psi$$

will not depend on θ. Putting

$$\frac{n_1+n_2}{2} - \frac{1}{2} = N$$

and substituting the explicit expression for $\rho(\xi,\eta)$ into this integral, we obtain the basic integral relation:

$$\iint_{\Omega} \psi(g(\xi,\eta)) \frac{\eta^{n_1-2}\, d\xi\, d\eta}{\left(\theta^2+\theta(1+\xi^2+\eta^2)+\eta^2\right)^N} = C_\psi \theta^{-n_1/2}(1+\theta)^{-N+1/2}, \quad (1)$$

where we may assume $C_\psi > 0$ for sufficiently large C.

Consider the right-hand side of equality (1) as a function of the complex variable $\theta = \sigma + it$. This function will be regular in the plane Π, cut along the negative axis: $-\infty < \sigma \leqslant 0$. The left-hand side of equality (1) will be regular in $\theta = \sigma + it$ in the same cut plane Π, if C is sufficiently large. The behavior of the left-hand side of (1) is especially transparent when n_1 and n_2 are numbers of different parity. In this case N is an integer, and in the integrand in (1) one can perform a decomposition into partial fractions.

Putting

$$A = A(\xi,\eta) = \frac{1+\xi^2+\eta^2}{2} - \sqrt{\left(\frac{1+\xi^2+\eta^2}{2}\right)^2 - \eta^2};$$

$$B = B(\xi,\eta) = \frac{1+\xi^2+\eta^2}{2} + \sqrt{\left(\frac{1+\xi^2+\eta^2}{2}\right)^2 - \eta^2},$$

we obtain:

$$\frac{1}{\left(\theta^2+\theta(1+\xi^2+\eta^2)+\eta^2\right)^N} = \frac{1}{(B-A)^N}\left(\frac{1}{(\theta+A)^N} + \frac{D_1}{(B-A)(\theta+A)^{N-1}} + \cdots \right.$$

$$\left. \cdots + \frac{D_{N-1}}{(B-A)^{N-1}(\theta+A)}\right) + \frac{1}{(A-B)^N}\left(\frac{1}{(\theta+B)^N} + \frac{E_1}{(A-B)(\theta+B)^{N-1}} + \cdots + \frac{E_{N-1}}{(A-B)^{N-1}(\theta+B)}\right),$$

where D_i, E_i depend only on N and i.

Substituting this expression under the integral sign in (1), we arrive at integral transformations similar to the Stieltjes transform (4). By virtue of the conditions imposed on $g(\xi,\eta)$, it is easy to see that for sufficiently large C the integral will converge absolutely and uniformly with respect to $\theta = \sigma + it$ in every bounded and closed region inside the cut plane Π.

Now consider the behavior of the right-hand side of (1) at the point

$$\theta = -1 + i\xi_0,$$

where $\xi_0 > 0$ is a small number. As $\xi_0 \to 0$, the asymptotic behavior of the right-hand side is clear. For further investigation of the left-hand side, we differentiate both sides of (1) with respect to the value C. The possibility of this is ensured by condition VII. The integral on the left-hand side turns into a sum of contour integrals along the level curves $g(\xi,\eta) = C$ (new integrand expressions then appear, involving $\operatorname{grad} g$ and the lengths of arcs of level lines, but not containing the parameter θ). The behavior of the integral on the left-hand side is determined by the mutual disposition of the level lines $g(\xi, \eta) = C$ and the curves where, for $\theta = -1 + i\xi_0$, small quantities occur in the denominators. These curves are named—

call the critical curves ("critics") of our family of measures, depending on the parameter θ. For the case under consideration there are two such "critics" :

$$A = A(\xi, \eta) = 1; \qquad B = B(\xi, \eta) = 1. \tag{2}$$

Moreover, the equality $A = B$ gives one more "critical point" :

$$\xi = 0, \qquad \eta = 1.$$

With the aid of properties I, . . . , VII, the contradiction in the behavior of the right- and left-hand sides as $\zeta_0 \to 0$ is detected rather simply. This proves the theorem.

The cumbersome conditions I, . . . , VII can be considerably simplified and weakened by means of a more detailed study of the mutual arrangement of the "critics" of the family of measures and the level lines of the test. The range of applicability of the proposed method may be quite substantial and applicable to a deepening of the least-squares method with unknown weights and of the analysis of variances. In addition to establishing the impossibility of certain similar tests, it may also be possible to apply it to the construction of approximately similar tests.

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REFERENCES CITED

[1] J. Neyman, Current Problems of Mathematical Statistics, report at the International Congress of Mathematicians, 1954, Amsterdam. [2] E. W. Barankin, Proc. I Berkeley Symposium on Prob. and Stat., Berkeley, 1949, 1950, p. 333. [3] A. Wald, Selected Papers, 1955. [4] D. Widder, The Laplace Transform, Princeton, 1946.

Note: Figure translations are in progress. See original paper for figures.

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