Reports of the Academy of Sciences of the USSR

R. F. BILYALOV

1963

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1963. Vol. 152, No. 3

MATHEMATICAL PHYSICS

R. F. BILYALOV

CONFORMAL GROUPS OF TRANSFORMATIONS IN GRAVITATIONAL FIELDS

(Presented by Academician V. A. Fock on 13 May 1963)

1. It is known how effective Lie group methods have proved to be in the theory of gravitational fields. In the present article conformal groups of transformations are considered as applied to gravitational fields. At first glance it seems that the addition of new functions on the right-hand sides of the Killing equations in determining conformal groups of transformations should lead to a broad extension of the classes of transformation groups admitted by Riemannian spaces. For gravitational fields, i.e. Riemannian spaces V_4 of signature $(---+)$, this extension turns out to be restrictively determined by the following theorem.

Theorem. *Conformal groups of transformations acting in Riemannian spaces V_4 that are not conformally flat are groups of motions or homotheties in spaces conformal to the given one.*

Therefore the classification of gravitational fields by groups of motions (([1], pp. 246-314)) is, in the main, also a classification by conformal groups of transformations.

The investigation is carried out in the class of analytic functions.

2. An r-parameter group of continuous transformations G_r with operators $X_a = \xi_a^\alpha p_\alpha$ is a group of conformal transformations of the Riemannian space V_n with metric $ds^2 = g_{ij}\,dx^i dx^j$, if the system of equations is satisfied (([2], p. 277))

$$\partial_\alpha g_{ij} + g_{i\beta}\partial_j \xi_f^\beta \eta_a^f + g_{j\beta}\partial_i \xi_f^\beta \eta_a^f = \psi_f \eta_a^f g_{ij}; \tag{1}$$

$$g_{i\alpha}\xi_f^\alpha \partial_j \varphi_s^f + g_{j\alpha}\xi_f^\alpha \partial_i \varphi_s^f = \Delta_s g_{ij}, \qquad \Delta_s = \psi_s - \varphi_s^f \psi_f; \tag{2}$$

$$\alpha, \beta, f, g = 1, \dots, q; \qquad i, j = 1, \dots, n; \qquad s = q+1, \dots, r,$$

where q is the rank of the matrix (ξ^α_a) (if $q < n$, then there exists a coordinate system in which $\xi^a_\sigma = 0$ $(\sigma = q{+}1, \dots, n)$ (([3], pp. 90-92))), for ξ^α_g in the equations $\det(\xi^\alpha_g) \neq 0$; φ^g_s are the coefficients of expansion with respect to the basis vectors of the remaining vectors; ψ_a are certain scalars; η^f_α is the reciprocal system with respect to the basis one. From (1) and (2), for ψ_a one obtains the system of equations

$$\partial_k \Delta_s = \eta^f_k (c^u_{fs} - \varphi^g_s c^u_{fg}) \Delta_u; \tag{3}$$

$$\partial_k \varphi_l - \partial_l \varphi_k = \eta^f_k \eta^g_l c^s_{fg} \Delta_s, \qquad \varphi_l = \eta^h_l \psi_h; \tag{4}$$

$$h, k, f, g, l = 1, \dots, q; \qquad s, u = q+1, \dots, r,$$

where c^c_{ab} are the structure constants of the group.

The complete system of integrability conditions for equations (1), (2), (3), and (4) is (2) and

$$(c^u_{ts} - \varphi^g_s c^u_{tg} - \varphi^g_t c^u_{sg} + \varphi^g_t \varphi^h_s c^u_{gh}) \Delta_u = 0; \tag{5}$$

$$t, s, u = q+1, \dots, r; \qquad g, h = 1, \dots, q.$$

3. For the metric $ds^{2'} = e^\alpha ds^2$ the group G_r becomes a group of conformal transformations with scalars $\psi'_a = \psi_a + \xi^k_a \partial_k \alpha$. When $\Delta_s \equiv 0$, the equations $\psi_a + \xi^k_a \partial_k \alpha = 0$ are completely integrable, and one has

Lemma 1. If $\Delta_s \equiv 0$, then there exists a space conformal to the given one, in which G_r is a group of motions.

From (2) it follows:

Lemma 2 for V_4 of signature $(- - - +)$**.** If the group G_r is nontransitive, and the surface of transitivity is nonisotropic or two-dimensional, then $\Delta_s \equiv 0$.

4. In the case of G_r acting on an isotropic surface of transitivity V^*_3 in V_4, in an isotropically semigeodesic coordinate system (([1], p. 269), where $g_{11} = g_{12} = g_{13} = 0$, $g_{14} = 1$, $g_{22}g_{33} - g^2_{23} > 0$, $\partial_1 \xi^2 = \partial_1 \xi^3 = 0$, $\psi = \partial_1 \xi^1$, $\xi^4 = 0$, passing in the integrability conditions (2) and (5) to a point, we obtain that the operators of the stationary subgroup in the space of zero-order operators will have the representation

$$
\begin{aligned}
&[X_1X_4] = 2X_1 && (\mathrm{mod}(X_4, \dots, X_r)),\\
&[X_2X_4] = X_2 - \alpha X_3 && (\mathrm{mod}(X_4, \dots, X_r)),\\
&[X_3X_4] = \alpha X_2 + X_3 && (\mathrm{mod}(X_4, \dots, X_r)),\\
&[X_1X_s] = 0, && (\mathrm{mod}(X_4, \dots, X_r)),\\
&[X_2X_s] = \beta_{sX}1 - \alpha_{sX}3 && (\mathrm{mod}(X_4, \dots, X_r)),\\
&[X_3X_s] = \gamma_{sX}1 + \alpha_{sX}2 && (\mathrm{mod}(X_4, \dots, X_r)),\\
&\quad c^4_{uv} = 0 \qquad (u, v = 4, \dots, r).
\end{aligned}
\tag{6}
$$

If the representation does not contain a matrix with diagonal elements different from zero, then, obviously, $\Delta_s \equiv 0$. The operators X_1, X_2, X_3 will belong to the roots 2, $1 \pm i\alpha$ of the characteristic polynomial of the group for the operator X_4 (([4], p. 239). There are no other roots whose real parts are greater than or equal to 2, since the operators corresponding to these roots would form an ideal in the minimal subalgebra containing these operators and the operators X_1, X_2, X_3, and would be operators of motion (([3], p. 260), which, as is seen from (6), belong to the roots 0, $1 \pm i\alpha$. Thus, the operators X_1, X_2, X_3 are operators of a subgroup of order 3 of the group G_r (([4], p. 243), and, for the admission of X_4 to be possible, for them reducibility to the forms

$$X_1 = p_1, \qquad X_2 = p_2, \qquad X_3 = p_3$$

or

$$X_1 = p_1, \qquad X_2 = p_2, \qquad X_3 = x^2 p_1 + p_3,$$

is necessary; and on the basis of Lemma 1 and (([1], pp. 273–274) the corresponding metrics may be taken in the form

$$
\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & g_{22} & g_{23} & 0\\ 0 & g_{32} & g_{33} & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & g_{22} & g_{23} & -x^3\\ 0 & g_{32} & g_{33} & 0\\ 1 & -x^3 & 0 & 0 \end{pmatrix}, \qquad \text{where } g_{ij} = g_{ij}(x^4).
$$

Then from (2) and from the presence in each subgroup of the operator $X_1 = p_1$ we obtain $\partial_1 \xi^1 = \psi = \mathrm{const}$, whence the validity of the theorem follows for the case under consideration.

5. For transitive conformal groups that are not groups of motions, the representation of the operators of the stationary subgroup in the space of zero-order operators will be given by the matrices

$$\begin{pmatrix} 1 & -\alpha_5 & -\beta_5 & a_5 \\ \alpha_5 & 1 & -\gamma_5 & b_5 \\ \beta_5 & \gamma_5 & 1 & c_5 \\ a_5 & b_5 & c_5 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -\alpha_s & -\beta_s & a_s \\ \alpha_s & 0 & -\gamma_s & b_s \\ \beta_s & \gamma_s & 0 & c_s \\ a_s & b_s & c_s & 0 \end{pmatrix} \qquad (s = 6, \dots, r), \quad (7)$$

$$c^5_{uv} = 0 \qquad (u, v = 5, \dots, r).$$

From (3) and (4) it is clear that, in order to prove the theorem, it suffices to prove that c^5_{ab}.

If all zero-order operators belong to roots of the characteristic polynomial of the group for the operator X_5 whose real parts are equal to 1, then the zero-order operators will form a commutative subalgebra and the space will be conformally flat.

For spaces that are not conformally flat, the representation (7) is exact. If Y is an operator to which the zero matrix corresponds, then for some i the operator $[X_{iY}]$ $(i = 1, \dots, 4)$ in the representation (7) must have a nonzero matrix in correspondence with it ((4), p. 203), but then the system of equations

$$[[X_{iY}]X_j] = [[X_{jY}]X_i] \pmod{(X_5, \dots, X_r)}, \qquad i, j = 1, \dots, 4,$$

has a nonzero solution only in the case when X_5 in (7) corresponds to the identity matrix.

With the aid of Lorentz transformations and linear transformations in the space of the operators $X_3,\ X_4$, we reduce the first matrix in (7) to the form

$$\begin{pmatrix} 1 & -\alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1+c & 0 \\ 0 & 0 & 0 & 1-c \end{pmatrix}, \qquad c > 0.$$

In order that the left product of two operators generate the operator X_5, it is necessary that the trace of the matrix corresponding to X_5 in the adjoint representation be equal to zero. The following matrix algebras satisfy this condition:

$$\begin{pmatrix} 1 & \alpha & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Only for the first two algebras is it possible that $[X_1 X_4] = aX_5$ or $[X_2 X_4] = bX_5$, but the latter contradict the Jacobi identities for the operators $X_1,\ X_2,\ X_3,\ X_4$.

In conclusion the author expresses deep gratitude to Prof. A. Z. Petrov and to the participants of his seminar for discussion of the results and valuable comments.

Kazan State University
named after V. I. Ulyanov-Lenin

Received
27 II 1963

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Note: Figure translations are in progress. See original paper for figures.