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Abstract

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SOME RESULTS OF THE ANALYSIS OF LARGE DEVIATIONS IN BOUNDARY PROBLEMS

(Presented by Academician A. N. Kolmogorov, January 29, 1963)

Let $s_0 = 0,\ s_1 = \xi_1,\ s_2 = \xi_1 + \xi_2, \ldots$ be a sequence of sums of independent identically distributed random variables $\xi_1, \xi_2, \ldots$, for which $M\xi_k = 0$. Let $x = x(n)$ be an increasing function of n, and let $g_1(t) > 0 > g_2(t)$ be two functions defined on the interval $[0, 1]$. In the well-known work of A. N. Kolmogorov (^{1}), and in a number of later works, the problem was considered of finding the asymptotic behavior, as $n \to \infty$, of the probability

$$\mathbf{P}\left(g_1\left(\frac{k}{n}\right) > \frac{s_k}{x} > g_2\left(\frac{k}{n}\right),\ k = 1, 2, \ldots, n\right). \tag{1}$$

The limiting value of (1) and estimates of the rate of convergence were found in the case when $x = \sqrt{n}$, and under very broad assumptions concerning the distribution of ξ_k (for bibliography see, for example, (^{2})).

The present note is devoted to probabilities of large deviations in problem (1) and related problems, when $x/\sqrt{n} \to \infty$. In this case the case of two boundaries g_1 and g_2 is easily reduced to the case of one boundary $g > 0$, while certain additional restrictions must be imposed on the random variables ξ_k. We shall consider the most meaningful "extreme" case, when

$$x = n(1 + \varkappa), \qquad \varkappa = o(1),$$

and assume that $\mathbf{P}(\xi_k < t)$ converges to its limits as $t \to \pm\infty$ according to exponential laws and contains a nonzero absolutely continuous component (the latter condition may be replaced by the lattice property of ξ_k). Introduce the necessary notation. Let η_g be the time of the first passage by the polygonal trajectory with vertices at the points $(k/n,\ s_k/x)$, $k = 0, 1, \ldots, n$, of the boundary g: $\eta_g = k/n$, if $s_j/x < g(j/n)$, $j < k$, $s_k/x \geq g(k/n)$; let $\chi_g = s_{n\eta_g} - xg(\eta_g)$ be the unnormalized amount of the first overshoot over g. Further, let $\varphi(\lambda) = Me^{-\lambda\xi_k}$, $\lambda_- = \inf\{\lambda : \varphi(\lambda) < \infty\}$, $\lambda_+ = \sup\{\lambda : \varphi(\lambda) < \infty\}$,

$$\mathfrak{m}(a) = \min_{\lambda \in [\lambda_-, \lambda_+]} e^{\lambda a} \varphi(\lambda),$$

let $\lambda(a)$ be the point of the interval $[\lambda_-, \lambda_+]$ at which this minimum is attained (it is unique), and

$$a_\pm = -\lim_{\lambda \uparrow\downarrow \lambda_\pm} \frac{\varphi'(\lambda)}{\varphi(\lambda)}.$$

The functions $\mathfrak{m}(a)$, $\lambda(a)$ are regular for $a \in (a_-, a_+)$, and $\lambda(a) = \lambda_\pm$ for $a \gtreqless a_\pm$.

We shall call problems of types I and II, respectively, the problems of finding the asymptotic behavior of the probabilities

$$\mathbf{P}\left(\eta_g = \frac{k}{n},\ \chi_g < y\right), \quad 1 \le k \le n; \qquad \mathbf{P}\left(\eta_g \le 1,\ \chi_g < y\right),$$

$$\mathbf{P}\left(\eta_g = \frac{k}{n},\ \chi_g < y,\ s_n < \rho x\right), \quad 1 \le k \le n; \qquad \mathbf{P}\left(\eta_g \le 1,\ \chi_g < y,\ s_n < \rho x\right).$$

The basic concept in the study of large deviations in boundary problems of types I and II is the notion of a level line.

The level line $a(t) > 0$ of type I (for problems of type I) is defined as a line along which the function $t \ln \mathfrak{m}\left(\frac{a}{t}\right)$, $0 < t \leqslant 1$, retains a constant value equal to $\ln \mathfrak{m}(\tau)$ (τ is a parameter). However, if $\alpha_- < \infty, \lambda_- = -\infty$ (ξ_k are bounded above by the value α_-) and $\mathfrak{m}(\alpha_-) = \mathbf{P}(\xi_k = \alpha_-) > 0$, then $\mathfrak{m}(\alpha)$ has a discontinuity at the point α_- (in this case $\mathfrak{m}(\alpha) = 0$ for $\alpha > \alpha_-$), and the function $a_\tau(t)$ is not defined for all t. For such t, by definition, we put $a_\tau(t) = \alpha_- t$. The line $a_\tau(t)$, $0 < t \leqslant 1,\ 0 < \tau < \infty$, is thus uniquely defined, increasing, convex, and is obtained by pasting together, generally speaking, two pieces of regular functions. The pasting occurs at the intersection of the curve $a_\tau(t)$ with the straight line $a/\alpha_- = t$. For values of t smaller than the abscissa of the pasting point, $a_\tau(t)$ is the continuation of the tangent to $a_\tau(t)$ at the pasting point.

The functions $a_\tau(t)$, where they differ from linear functions, can also be defined as the solution $a_\tau(t) = a(\tau, t)$ of the Cauchy problem for the absolute equation

$$a''_{\tau t}\left(\frac{a}{t} - a'_t\right) + a''_{tt} a'_\tau = 0$$

under the conditions $a(\tau, 1) = \tau,\ a'_t(\tau, 1) = -\ln \varphi(\lambda(\tau))/\lambda(\tau)$.

Next, we shall say that a function $R(t)$, defined in a neighborhood δ of the point $t = 0$, admits a power majorant if there exist positive numbers r, c_1, c_2 and a function $r(t) \downarrow 0$ as $|t| \downarrow 0$ such that $|R(t)| < c_1 r(t)$ for $t \in \delta$ and $r(td) \geqslant c_2 d^r r(t)$ for all $0 < d \leqslant 1$ and $t \in \delta$.

Theorem 1. *Suppose the following conditions are fulfilled:*

1) $k/n \to v$ as $n \to \infty$, v does not depend on n and belongs to the semi-interval $(0, 1]$.

2) The function $g(t)$ in a neighborhood of the point v has $p \geqslant 1$ derivatives. In the representation

$$g(t) = g(u) + \dots + (t-u)^p \frac{g^{(p)}(u)}{p!} + (t-u)^p R_u^1(t-u)$$

the remainder factor $R_u^1(t-u)$, for all u, $|u-v| < \varepsilon_1$, admits one and the same power majorant.

3) $g'(v) < g(v)/v < \alpha_-$.

4) For any $\varepsilon_2 > 0$ there exists $\varepsilon_3 > 0$ such that $g(t) > \dfrac{g(v)}{v} t + \varepsilon_3$ for all $t < v - \varepsilon_2$.

Then, for any fixed $y > 0$,

$$\mathbf{P}\left(\eta_g = \frac{k}{n},\ \chi_g < y\right) =$$

$$= \frac{\mathfrak{m}^n(\tau_\sigma)}{\sqrt{k}}\, \Psi\left(\frac{g(v)}{v},\, g'(v),\, y\right) e^{nT(\Delta,\sigma,v)} \left\{1 + O\left(\frac{1}{n}, \Delta, \varkappa, r_2\left(\frac{1}{n}\right), R_v^3(\Delta)\right)\right\}. \tag{2}$$

Here τ_σ is the solution of the equation $a_\tau(v) = \sigma g(v)$, $\sigma = \varkappa/n = 1 + \varkappa$, $\Delta = k/n - v$.

$r_2(s)$ is a majorant of the function

$$R_u^2(s) = \frac{g(u+s) - g(u) - s g'(u)}{s}, \qquad R_v^3(s) = g'(v+s) - g'(v),$$

$$T(t-v, \sigma, v) = T_0(t) + \sum_{k=1}^{\infty} \varkappa^k \left[T_k(t) - T_k(v)\right],$$

$$T_0(t) = t \int_{a_{\tau_1}(t)/t}^{g(t)/t} \lambda(\alpha)\, d\alpha, \qquad T_k(t) = \frac{g(t)}{k!}\, \lambda^{(k-1)}\left(\frac{g(t)}{t}\right) \left(\frac{g(t)}{t}\right)^{k-1}, \quad k = 1, 2, \dots$$

The function $\Psi\left(g(v)/v, g'(v), y\right)$ depends only on its arguments and on the distribution of the ξ_k and can be written in the notation of theorem 5 from [3]. The estimate in (2) is uniform in k for $|k/n - v| < \varepsilon_1$.

Let the following quantity be defined and finite:

$$\tau_g = \sup\left\{\tau : \inf_{0<t<1}\left(g(t) - \mathfrak{a}_\tau(t)\right) > 0\right\}$$

and let V_ε be the set of points of the interval $[0,1]$ for which

$$g(t) \leqslant \mathfrak{a}_{\tau_g}(t) + \varepsilon,$$

and let V^g be the interval

$$[0, \ln \mathfrak{m}(\tau_g)/\ln \mathfrak{m}(\alpha_-)].$$

Suppose that for sufficiently small ε the intersection $V_\varepsilon \cap V^g$ is empty. Finally, let Ω be the union of all intervals belonging to V_0, and let $\bar{A}$ denote the closure of the set A. For $x = n$ the following holds.

Theorem 2. *If, in addition to the assumptions made, the Lebesgue measure*

$$\operatorname{mes}(\bar{V}_\varepsilon - \Omega) \to 0 \quad \text{as } \varepsilon \to 0,$$

then

$$\mathbf{P}(\eta_g \leqslant 1,\ \chi_g < y) \sim \sqrt{n\mathfrak{m}^n(\tau_g)} \int_\Omega t^{-1/2}\Psi\left(\frac{g(t)}{t},\, g'(t),\, y\right) dt.$$

Now let $\operatorname{mes} V_0 = 0$, and let V_ε, for all ε smaller than some number, be a finite number of half-intervals or intervals on which g is piecewise differentiable. In this case the asymptotic behavior of the probability

$$\mathbf{P}(\eta_g \leqslant 1,\ \chi_g < y)$$

is clarified by the following theorem.

Theorem 3. *Let the set V_ε, for all $\varepsilon > 0$ smaller than some number, consist of one half-interval or interval, and let the set*

$$\bigcap_{k=1}^{\infty} \bar{V}_{1/k}$$

consist of a single point v, which is a boundary point of V_ε. Suppose, further, that in the half-neighborhood V_ε of the point v condition 2 of Theorem 1 is satisfied. Then, if the number q of common first derivatives of the functions g and $\mathfrak{a}_{\tau_g}$ at the point v is positive and $p \geqslant q + 1$, then

$$\mathbf{P}(\eta_g \leqslant 1,\ \chi_g < y) =$$

$$= n^{q-1/2q+2}\mathfrak{m}^n(\tau_g)\frac{1}{q+1}\Gamma\left(\frac{1}{q+1}\right)\left[\frac{\lambda(g(v)/v)(\mathfrak{a}_{\tau_g}^{(q+1)}(v) - g^{(q+1)}(v))}{(q+1)!}\right]^{-1/(q+1)} \times$$

$$\times\Psi\left(\frac{g(v)}{v},\, g'(v),\, y\right)\left(1 + O\left(n^{-1/(q+1)},\; r_T(n^{-1/(q+1)})\right)\right).$$

Here, by derivatives of the functions g at the point v, one means respectively right- or left-hand derivatives; $r_T(t-v)$ is the power majorant of the function

$$\frac{1}{(t-v)^{q+1}}\left\{t\int_{\mathfrak{a}_{\tau_g}(t)/t}^{g(t)/t}\lambda(\alpha)\,d\alpha - (t-v)^{q+1}\frac{\lambda(g(v)/v)(g^{(q+1)}(v) - \mathfrak{a}_{\tau_g}^{(q+1)}(v))}{(q+1)!}\right\}.$$

It is also possible to analyze less regular cases, when $p = q$ or $q = 0$, etc. As is clear from Theorems 2 and 3, the asymptotics of probability (1) in the case of large deviations is determined only by the properties of the function g on the sets V_ε and by the properties of the sets V_ε themselves.

Level lines in problems of type II are defined differently. Denote

$$\mathfrak{m}(\tau,\rho,t) = t\ln\mathfrak{m}\left(\frac{\tau}{t}\right) + (1-t)\ln\mathfrak{m}\left(\frac{\rho-\tau}{1-t}\right),$$

$$\mathfrak{m}(\tau,\rho) = \sup_{0<t<1}\mathfrak{m}(\tau,\rho,t),$$

and suppose that τ and ρ, $\tau > \rho$, are such that

$$\mathfrak{M}(\tau,\rho) > -\infty.$$

We first define the level lines $\mathfrak{b}_{\tau,\rho}(t)$ of the so-called local problem as the solution of the equation

$$\mathfrak{M}(\mathfrak{b},\rho,t) = \mathfrak{m}(\tau,\rho), \qquad 0 < t < 1.$$

In those cases where $\mathfrak{m}(\alpha)$ has a discontinuity at the points α_+ or α_-, the solution may fail to exist for some t. In these regions we complete the level lines (as also the level lines of type I) by line segments

$$\mathfrak{b} = \alpha\_t$$

or

$$\mathfrak{b} = \rho - \alpha_+(1-t).$$

This definition of the functions $\mathfrak{b}_{\tau,\rho}(t)$ will be unique. The level lines $\mathfrak{b}_{\tau,\rho}(t)$ themselves turn out to be convex, continuously differentiable, and glued together from pieces of, generally speaking, three regular functions. Let D_1—D_4 be the four sectors into which the domain $\mathfrak{b} \geq \rho t,\ 0 \leq t \leq 1$, is divided by the straight lines $\mathfrak{b} = \alpha\_t$ and $\mathfrak{b} = \rho - \alpha_+(1-t)$. Then $\mathfrak{b}_{\tau,\rho}(t)$ is regular in each of these sectors, on the boundaries of which the gluing also takes place. If t_0 is the point at which $\sup_{0<t<1}\mathfrak{M}(\tau,\rho,t)$ is attained, then

$\mathfrak{b}_{\tau,\rho}(t_0) = \tau$, $\mathfrak{b}_{\tau,\rho}(t) \leq \tau$ for any $t \in (0,1)$. In the domain of regularity $\mathfrak{b}_{\tau,\rho}(t)$ increases with increasing τ.

The required level lines of type II are obtained from the level lines of the local problem by replacing the part of the curve $\mathfrak{b}_{\tau,\rho}(t)$ lying in the domain $0 < \mathfrak{b} < \rho$ (empty when $\rho \leq 0$) by the part of the curve $\mathfrak{c}_{\tau,\rho}(t)$ determined by the solution of the equation

$$t \ln \mathfrak{m}\left(\frac{c}{t}\right) = \mathfrak{m}(\tau, \rho),$$

i.e., by a part of a level line of type I. We denote the newly obtained curves by $\mathfrak{d}_{\tau,\rho}(t)$.

Now we can formulate some assertions concerning problems of type II. Put again $x = n$,

$$\tau_g = \sup\left\{\tau : \inf_{0<t<1}(g(t) - \mathfrak{d}_{\tau,\rho}(t) > 0)\right\}$$

and denote by V_ε the set of points of the interval $[0,1]$ for which $g(t) \leq \mathfrak{d}_{\tau_g,\rho}(t)+\varepsilon$, and by V^g the closure of the set of those t for which the point $(t, \mathfrak{d}_{\tau_g,\rho}(t))$ does not belong to the lower of the sectors D_1—D_4. We shall again assume that, for sufficiently small ε, the intersection $V_\varepsilon \cap V^g$ is empty.

Theorem 4. Let Ω be the union of all intervals of the set V_0. If $\mathrm{mes}(V_\varepsilon - \Omega) \to 0$ as $\varepsilon \to 0$, then

$$\mathbf{P}(\eta_g \leq 1,\ \chi_g < y,\ s_n < \rho n) \sim e^{n\mathfrak{m}(\tau_g,\rho)} \int_\Omega \Phi(t,y)\, dt,$$

where

$$\Phi(t,y) = \frac{W(\gamma(t))}{\sqrt{t(1-t)}} \int_0^y e^{-y\lambda(\gamma(t))}\, d_y \Psi\left(\frac{g(t)}{t},\, g'(t),\, y\right),$$

$$\gamma(t) = \frac{\rho - g(t)}{1-t}, \qquad W(\gamma) = \frac{1}{\lambda(\gamma)\sqrt{2\pi\varphi''_\lambda(\lambda(\gamma),\gamma)/\varphi(\lambda(\gamma),\gamma)}}, \qquad \varphi(\lambda,\gamma) = e^{\gamma\lambda}\varphi(\lambda).$$

If g is tangent to $\mathfrak{d}_{\tau_g,\rho}$ at a finite number of points, then the desired value can be computed by using the following analogue of Theorem 3.

Theorem 5. If, in the notation introduced anew, the conditions of Theorem 3 are satisfied, then

$$\mathbf{P}(\eta_g \leq 1,\ \chi_g < y,\ s_n < \rho n) \sim$$

$$\sim n^{-1/(q+1)} e^{n\mathfrak{M}(\tau_g,\rho)} \Phi(\nu, y) \frac{1}{q+1} \Gamma\left(\frac{1}{q+1}\right) U^{-1/(q+1)},$$

where

$$U = \frac{[\mathfrak{d}_{\tau_g,\rho}^{(q+1)}(\nu) - g^{(q+1)}(\nu)][\lambda(g(\nu)/\nu) - \lambda((\rho - g(\nu))/(1-\nu))]}{(q+1)!}.$$

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Note: Figure translations are in progress. See original paper for figures.