G. KARATOPRAKLIEV

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Abstract

Full Text

G. KARATOPRAKLIEV

ON A GENERALIZATION OF PROBLEM T FOR THE EQUATION

$$u_{xx} + \operatorname{sign} y\, u_{yy} = 0$$

(Presented by Academician M. A. Lavrent' ev, 27 IV 1963)

In the present article one boundary-value problem is considered for the Lavrent' ev—Bitsadze equation

$$u_{xx} + \operatorname{sign} y\, u_{yy} = 0, \tag{1}$$

which is a generalization of the Tricomi problem T, posed and investigated in the work of M. A. Lavrent' ev and A. V. Bitsadze ([1]) and in the works of A. V. Bitsadze ([2], [3]).

Let D be a simply connected finite domain of the xy-plane, bounded by a Jordan line σ with endpoints at the points $A(-1, 0)$, $B(1, 0)$, lying in the upper half-plane $y > 0$, and by the characteristics $AC : y = -x - 1$ and $BC : y = x - 1$, issuing from the point $C(0, -1)$. Denote by D_1 and D_2, respectively, the elliptic and hyperbolic parts of the mixed domain D.

Problem T_α. It is required to determine a function $u(x, y)$ with the following properties: 1) $u(x, y)$ is a solution of equation (1) in the domain D for $y \neq 0$; 2) $u(x, y)$ is continuous in the closed domain D for $y \neq 0$; 3) the partial derivatives u_x and u_y are continuous in the domain D for $y \neq 0$, and near the points A and B they may become infinite of order less than one; 4) on the segment AB of the real axis the functions $u(x, y)$ and $u_y(x, y)$ satisfy the gluing conditions

$$u(x, +0) = \alpha(x)u(x, -0), \tag{2}$$

$$u_y(x, +0) = \beta(x)u_y(x, -0), \tag{3}$$

where $\alpha(x)$ and $\beta(x)$ are given functions which do not vanish anywhere on the segment AB, $\beta(x)$ is once and $\alpha(x)$ twice differentiable, and the functions $\beta'(x)$ and $\alpha''(x)$ belong to the class $\bar{H}$ on the segment ${}^*AB$, with

$$\alpha'(x)\beta(x) \leq 0, \qquad -1 < x < 1; \tag{4}$$

5) $u(x, y)$ assumes the prescribed values

$$u = \varphi \qquad \text{on } \sigma, \tag{5}$$

$$u = \psi(x) \qquad \text{on } AC, \tag{6}$$

where φ is continuous, and $\psi(x)$ is a twice differentiable function whose second derivative belongs to the class $\bar{H}$ on the segment $[-1,0]$, and $\varphi(-1) = \alpha(-1)\psi(-1)$.

For $\alpha(x) = \beta(x) = 1$, problem T_α coincides with problem T.

In the domain D_2, by virtue of (2) and (3), the solution $u(x,y)$ of equation (1) has the form

$$2u(x,y) = \frac{\tau(x+y)}{\alpha(x+y)} + \frac{\tau(x-y)}{\alpha(x-y)} + \int_{x-y}^{x+y} \frac{\nu(t)}{\beta(t)}\,dt, \tag{7}$$

where $\tau(x) = u(x,+0)$, $\nu(x) = u_y(x,+0)$, $-1 \le x \le 1$.

By virtue of (6), from (7) we obtain

$$\frac{\tau(x)}{\alpha(x)} + \frac{\tau(-1)}{\alpha(-1)} - \int_{-1}^{x} \frac{\nu(t)}{\beta(t)}\,dt = 2\psi\left(\frac{x-1}{2}\right), \qquad -1 \le x \le 1,$$

* We use the terminology adopted in ([6]).

or, after differentiation,

$$\left[\frac{\tau(x)}{\alpha(x)}\right]' - \frac{\nu(x)}{\beta(x)} = 2\frac{d}{dx}\psi\left(\frac{x-1}{2}\right), \qquad -1 < x < 1. \tag{8}$$

Hence, taking (4) into account, just as in problem T ([3]), we conclude that if $\psi(x) \equiv 0$, the solution $u(x,y)$ of problem T_α in the closed domain $\overline{D}_1$ attains a nonzero extremum on the arc σ (the extremum principle). From this principle the uniqueness of the solution of problem T_α follows directly.

The existence of a solution of problem T_α will be proved if the function $\nu(x)$ can be determined. Let us note that, without loss of generality, one may assume that $\varphi = 0$. We shall additionally assume that σ is a smooth arc satisfying the Lyapunov condition, and that u_x and u_y are continuous in the closed domain $\overline{D}_1$ everywhere except, possibly, at the points A and B. By a conformal mapping one can arrange that σ coincide with the semicircle σ_0 with endpoints at the points A and B ([3]). We shall assume that σ coincides with σ_0.

The relation between the functions $\tau(x)$ and $\nu(x)$ from the elliptic part D_1 of the mixed domain D has the form ([3])

$$\tau(x) - \frac{1}{\pi}\int_{-1}^{1} [\ln|t-x| - \ln(1-tx)]\nu(t)\,dt = 0, \qquad -1 \le x \le 1. \tag{9}$$

We shall seek the function $\nu(x)$ in the class H^* on the interval $[-1,1]$. Eliminating $\tau(x)$ from (8) and (9), to determine $\nu(x)$ we obtain the singular integral equation

$$\alpha(x)\nu(x) + \frac{\beta(x)}{\pi}\int_{-1}^{1}\left(\frac{1}{t-x} - \frac{t}{1-tx}\right)\nu(t)\,dt +$$

$$+\alpha'(x)\beta(x)\int_{-1}^{1}\frac{\omega(x,t)}{\beta(t)}\nu(t)\,dt = f(x), \tag{10}$$

where

$$f(x) = -2\beta(x)\left[\alpha(x)\psi\left(\frac{x-1}{2}\right)\right]',$$

and $\omega(x,t) = 1$ for $t \in [-1, x]$, $\omega(x,t) = 0$ for $t \notin [-1, x]$.

Equation (10) can very simply be reduced to a Fredholm equation. For this purpose we rewrite it in the form

$$\alpha(x)\nu(x) + \frac{\beta(x)}{\pi}\int_{-1}^{1}\left(\frac{1}{t-x} - \frac{t}{1-tx}\right)\nu(t)\,dt = g(x), \tag{11}$$

where

$$g(x) = f(x) - \alpha'(x)\beta(x)\int_{-1}^{1}\frac{\omega(x,t)}{\beta(t)}\nu(t)\,dt, \tag{12}$$

and we shall regard $g(x)$ (belonging to the class H) as a known function.

Equation (11) is associated with the cyclic group $\mathcal{W}_0(z) = z$, $\mathcal{W}_1(z) = 1/z$, and therefore is solvable explicitly ([3,4]). We note only the following: since the contour of integration is open and the transformation $\mathcal{W}_1(z) = 1/z$, taking the interval $[-1, 1]$ into the remaining part of the real axis, leaves the endpoints fixed, then, as is easy to see, one cannot seek a solution of class h_0 of equation (11). The index of this equation is equal to zero.

For the function $v(x)$ we obtain the expression

$$v(x) = a(x)g(x) - \frac{b(x)Z(x)}{\pi}\int_{-1}^{1}\frac{g(t)}{Z(t)}\left(\frac{1}{t-x} - \frac{t}{1-tx}\right)dt, \tag{13}$$

where

$$a(x) = \frac{\alpha(x)}{\alpha^2(x)+\beta^2(x)}, \qquad b(x) = \frac{\beta(x)}{\alpha^2(x)+\beta^2(x)}, \qquad Z(x) = \sqrt{\alpha^2(x)+\beta^2(x)}\, e^{\Gamma(x)},$$

$$\Gamma(x) = \int_{-1}^{1} \theta(t) \left[\frac{1}{t-x} - \frac{1}{t(1-tx)} \right] dt, \qquad \theta(x) = -\frac{1}{\pi} \operatorname{arc\,tg}_{(-\pi/2,\ \pi/2)} \frac{\beta(x)}{\alpha(x)}.$$

Taking into account (12), for $v(x)$ we obtain a Fredholm integral equation equivalent to equation (10):

$$v(x) + \int_{-1}^{1} K(x,t) v(t)\, dt = h(x), \tag{14}$$

where

$$K(x,t) = \frac{a(x)\alpha'(x)\beta(x)\omega(x,t)}{\beta(t)} - \frac{b(x)Z(x)}{\pi\beta(t)} \int_{-1}^{1} \frac{\alpha'(t_1)\beta(t_1)\omega(t_1,t)}{Z(t_1)} \left(\frac{1}{t_1 - x} - \frac{t}{1 - t_1 x} \right) dt_1,$$

$$h(x) = a(x) f(x) - \frac{b(x)Z(x)}{\pi} \int_{-1}^{1} \frac{f(t)}{Z(t)} \left(\frac{1}{t-x} - \frac{t}{1-tx} \right) dt.$$

All the basic Fredholm theorems are applicable to equation (14) [5,6]. From the uniqueness of the solution of problem T_α it follows that equation (14) is solvable. It is not hard to see that the function $v(x)$, which is the solution of equation (14), belongs to the class H^* on the interval $[-1,1]$ and is differentiable in the interval $(-1,1)$.

It is interesting to note that if $\beta(x)/\alpha(x) < 0$, $-1 \leq x \leq 1$, the solution $v(x)$ tends to infinity of order less than one at $x = -1$; if $\beta(x)/\alpha(x) > 0$, $-1 \leq x \leq 1$, then, as is to be expected, $v(x)$ tends to infinity of order less than one at $x = 1$.

Corollary. If $\alpha(x) = \text{const}$, condition (4) is satisfied, $K(x,t) = 0$, and the solution $v(x)$ of equation (10) is found explicitly:

$$v(x) = \frac{\alpha(x) f(x)}{\alpha^2(x) + \beta^2(x)} - \frac{\beta(x) e^{\Gamma(x)}}{\pi\sqrt{\alpha^2(x) + \beta^2(x)}} \int_{-1}^{1} \frac{f(t) e^{-\Gamma(t)}}{\sqrt{\alpha^2(x) + \beta^2(x)}} \left(\frac{1}{t-x} - \frac{t}{1-tx} \right) dt. \tag{15}$$

Let us note that in this case, in order to prove the existence of a solution of problem T_α, one can use the results of [7], if equation (1) is reduced to a system of mixed type.

For $\alpha(x) = \beta(x) = 1$, from formula (15) we obtain the function $v(x)$ for problem T.

Mathematical Institute with Computing Center
of the Bulgarian Academy of Sciences

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Note: Figure translations are in progress. See original paper for figures.