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Abstract

Full Text

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On Properties of the Spectra of Ergodic Dynamical Systems

(Presented by Academician A. N. Kolmogorov, 18 I 1963)

Below a number of theorems on properties of the spectra of ergodic dynamical systems are formulated.

It is known ([1,2]) that the spectrum of an ergodic dynamical system, in the case when it is discrete, has the following properties:

1. The totality of all eigenvalues forms a countable subgroup of the additive group of real numbers in the case of systems with continuous time, or of the multiplicative group of numbers of modulus 1 in the case of discrete time.

2. The multiplicity of each eigenvalue is equal to one.

3. The modulus of each eigenfunction is almost everywhere constant.

Let the dynamical system $\{S_t\}$ be a one-parameter group of measure-preserving transformations of some space with measure $(M, \mathfrak{S}, \mu)$. The parameter t runs either through the integers or through all real values. The one-parameter group of unitary operators conjugate to $\{S_t\}$, acting in the Hilbert space $\mathcal{L}^2_\mu(M)$, is denoted by $\{U_t\}$, and the corresponding spectral family of projection operators by $\{E(\Delta)\}$, where Δ, in the case of continuous time, is a Borel subset of the line, and in the case of discrete time, a Borel subset of the unit circle.

§ 1. The following definition is a transfer to the case of dynamical systems of the definition introduced by Fortet and Blanc-Lapierre for stationary processes ([3]).

Definition (cf. ([3,4])). An element $f \in \mathcal{L}^2_\mu(M)$ belongs to the class $F^{k,l}$, if for any $k_1, l_1,\ k_1 \leq k,\ l_1 \leq l$, in the $(k_1 + l_1)$-dimensional space

$$(\lambda_1, \dots, \lambda_{k_1}, \lambda'_1, \dots, \lambda'_{l_1})$$

there exists such a countably additive function of sets of bounded variation $M^{k_1 l_1}$, that for any $(k_1 + l_1)$-measurable subsets of the axis λ,

$$\Delta_1, \dots, \Delta_{k_1}, \Delta'_1, \dots, \Delta'_{l_1}$$

and the parallelepiped

$$\Delta = \Delta_1 \times \dots \times \Delta_{k_1} \times \Delta'_1 \times \dots \times \Delta'_{l_1}$$

in the $(k_1 + l_1)$-dimensional space

$$M^{k_1 l_1}(\Delta) = \int_M E(\Delta_1) f \dots E(\Delta_{k_1}) f \cdot \overline{E(\Delta'_1) f} \dots \overline{E(\Delta'_{l_1}) f} \, d\mu.$$

It is known that $M^{k_1 l_1}$, by virtue of the invariance of the measure μ, must be concentrated on the hyperplane

$$\lambda_1 + \dots + \lambda_{k_1} - \lambda'_1 - \dots - \lambda'_{l_1} = 0$$

(see ([3,4])).

In what follows the following condition is assumed to be satisfied.

Condition A. The set of vectors f of the Hilbert space $\mathcal{L}^2_\mu(M)$ belonging to the class $F^{2,2}$ is everywhere dense in $\mathcal{L}^2_\mu(M)$.

Condition A is essentially a requirement on the rate of multiple mixing of the system $\{S_t\}$. It is not difficult to give fairly general conditions under which a dynamical system, representable as a factor-system of a dynamical

system generated by a finite-dimensional Gaussian stationary process will satisfy condition A. Apparently, in general, a fairly broad class of dynamical systems satisfies condition A. It is not excluded that all K-systems satisfy condition A. At the same time one can indicate examples of dynamical systems that do not satisfy this condition. A. N. Kolmogorov pointed out the existence of examples of this kind (see (5)). We shall give here an example arising from dynamical systems with quasidiscrete spectrum, studied by P. Halmos and von Neumann (2), L. M. Abramov (6), and others.

Let θ_1 and θ_2 be independent random variables uniformly distributed on the interval $[0, 1]$. Take the sequence of random variables

$$\xi_n = \exp 2\pi i \left[\frac{(n-1)n}{2} \alpha + n\theta_1 + \theta_2 \right],$$

where α is an irrational number. It is not difficult to verify that this sequence of random variables forms an ergodic stationary process (for this it is convenient to pass to the torus T^2 and its transformation $A : (\theta_1, \theta_2) \to (\theta_1 + \alpha \pmod 1), \theta_1 + \theta_2 \pmod 1))$, and then show that $\xi_n(\theta_1, \theta_2) = \xi_0(A^n(\theta_1, \theta_2))$; $\xi_0(\theta_1, \theta_2) = \exp 2\pi i \theta_2)$. A direct computation shows that the dynamical system generated by this process does not satisfy condition A. In particular, the element of the Hilbert space $\xi_0(\theta_1, \theta_2)$, being bounded, does not belong to the class F^{22}. It is possible that all dynamical systems with quasidiscrete spectrum do not satisfy condition A.

§ 2. For dynamical systems satisfying condition A, one can establish a number of properties of their spectra which it is natural to regard as analogues of the spectral properties listed above for dynamical systems with discrete spectrum.

Theorem 1. *Let $d\sigma(\lambda)$ be a finite measure whose type coincides with the maximal spectral type of an ergodic system $\{S_t\}$ satisfying condition A. Then*

$$\sigma * \sigma \sim \sigma, \qquad \bar{\sigma} \sim \sigma, \tag{1}$$

where $*$ is convolution, $\sim$ is the sign of equivalence of measures, $\bar{\sigma}(d\lambda) = \sigma(-d\lambda)$.

It should be noted that in the case under consideration $\sigma(\{0\}) > 0$. The latter property, indicated in (1), has been known for a very long time. Theorem 1 is naturally regarded as a generalization of the fact that, in the case of a discrete spectrum, the set of eigenvalues forms a group. Let $\sigma_e(\Delta) = \sigma(\{0\} \cap \Delta)$, $\sigma_r(\Delta) = \sigma(\Delta) - \sigma_e(\Delta)$.

Theorem 1′. *Let $d\sigma(\lambda)$ be such that $\sigma_r(\Delta)$ has no atoms of positive measure, i.e. constants are the only eigenfunctions of the group $\{U_t\}$. Then there exists an ergodic dynamical system whose maximal spectral type coincides with the type of the measure $d\sigma$.*

Theorem 2. *Let the element $f \in F^{22}$ and $\Delta = \{\lambda : \lambda_1 \leq \lambda < \lambda_2\}$,*

$$\Delta_n^i = \left\{\lambda : \lambda_1 + \frac{i}{2^n}(\lambda_2 - \lambda_1) \leq \lambda < \lambda_1 + \frac{i+1}{2^n}(\lambda_2 - \lambda_1)\right\}, \quad i = 0, 1, \dots, 2^n - 1.$$

Then

$$\lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left|E(\Delta_n^i) f\right|^2 = (E(\Delta) f, f),$$

where the limit on the left is understood in the mean-square sense.

Theorem 2 is naturally regarded as an analogue of the fact that, in the case of a discrete spectrum, the eigenfunctions are almost everywhere constant in modulus.

§ 3. In many studied examples of dynamical systems with continuous spectrum, the spectrum turned out to be multiple and even, as a rule, of infinite multiplicity. The construction of examples of systems with simple or finite-multiple

with continuous spectrum requires very special techniques and was carried out in the work of I. V. Girsanov (7).

We now give a theorem showing that, in the class of systems satisfying condition A, a finite-multiplicity spectrum requires very special properties of the maximal spectral type.

Let $d\sigma(\lambda)$ be a finite measure on the line, and let $d\sigma(\lambda_1) \cdot d\sigma(\lambda_2)$ be the direct product of this measure with itself, giving a measure on the plane. Then one can write

$$d\sigma(\lambda_1) d\sigma(\lambda_2) = d\mu(\alpha) \cdot d\nu(\beta/\alpha); \tag{2}$$

where $\alpha = \lambda_1 + \lambda_2$, $\beta = \lambda_1 - \lambda_2$, $\mu = \sigma * \sigma$, ν is, for almost every α, a measure in β ("conditional measure"), and the measures ν are connected with μ by known measurability conditions.

Theorem 3. Let σ be a measure whose type coincides with the maximal spectral type of an ergodic system $\{S_t\}$ satisfying condition A. Let Δ be the set of those α for which the corresponding measure $d\nu(\beta/\alpha)$ in (2) has a set of positive measure on which it is continuous, and let $\sigma_\Delta(\Delta_1) = \sigma(\Delta \cap \Delta_1)$. Then the multiplicity of the spectral type of the measure σ_Δ is infinite.

Corollary 1. In order that, for an ergodic dynamical system $\{S_t\}$ satisfying condition A, the spectral multiplicity function be almost everywhere finite,* it is necessary that, for some (and hence also for every) measure $d\sigma$ whose type coincides with the maximal spectral type of the system $\{S_t\}$, the measures ν in the decomposition (2) for σ be, for almost every α (with respect to $d\mu = d(\sigma * \sigma)$), discrete, i.e. concentrated at no more than a countable number of points.

If some measure $d\rho(\lambda) = f(\lambda)\, d\lambda$, absolutely continuous with respect to Lebesgue measure, is such that $d\rho \ll d\sigma$, then we shall say that the type of the measure $d\sigma$ dominates the absolutely continuous type of the measure $d\rho$. If $f(\lambda) > 0$ almost everywhere with respect to Lebesgue measure, then $d\rho(\lambda)$ is equivalent to Lebesgue measure and, consequently, has Lebesgue type.

Corollary 2. If the maximal spectral type of an ergodic system $\{S_t\}$ satisfying condition A dominates some absolutely continuous type, then it also dominates the Lebesgue type, and the Lebesgue type has countable multiplicity.

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* Finiteness of the multiplicity function is equivalent to the fact that each homogeneous spectral type has finite multiplicity.

Note: Figure translations are in progress. See original paper for figures.