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Abstract

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REMAINDER TERMS IN THE PROBLEM OF THE DISTRIBUTION OF VALUES OF TWO ARITHMETIC FUNCTIONS

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Let $f(n)$ denote some real-valued function of a natural argument, and let λ be a real number. By $P_N(f(n) < \lambda)$ we shall denote the number of integers among $1, 2, \dots, N$ for which $f(n) < \lambda$.

1. Let $f(n) = \varphi(n)/n$, where $\varphi(n)$ is Euler's function. Schoenberg ([1]) proved that for every real λ there exists the limit

$$\lim_{N\to\infty} \frac{P_N(f(n) < \lambda)}{N} = v(\lambda), \tag{1}$$

and the limiting function is continuous. The properties of the function $v(\lambda)$ are the subject of a paper by B. A. Venkov ([2]).

Since the values of $\varphi(n)/n$ are contained in the interval $[0, 1]$, it follows, by the second theorem of Helly (([3]), p. 220), that for every function $F(x)$ continuous on the interval $[0, 1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F\left(\frac{\varphi(n)}{n}\right) = \int_0^1 F(x)\, dv(x). \tag{2}$$

We shall formulate theorems concerning the remainder terms in equalities (1) and (2).

Theorem 1. Let λ belong to the interval $[0, 1]$. Then

$$\frac{P_N(\varphi(n)/n < \lambda)}{N} = v(\lambda) + O\left(\frac{1}{\ln\ln\ln N}\right).$$

Let $F(x)$ be defined on $[0, 1]$. Denote $\max_{0\le x\le 1} |F(x)| = M$.

Theorem 2. Given an integer function of order ρ, $F(x)$. For every $\varepsilon > 0$,

$$\frac{1}{N}\sum_{n\le N}F\left(\frac{\varphi(n)}{n}\right)=\int_0^1 F(x)\,dv(x)+O\left(\frac{M+1}{N^{\frac{1}{1+\rho}-\varepsilon}}\right).$$

Theorem 3. Let $F(x)$ be a function given on the interval $[0,1]$. Suppose that the function is analytic inside the ellipse with foci at the points 0 and 1 and with sum of semiaxes equal to R ($R>1/2$). Then

$$\frac{1}{N}\sum_{n\le N}F\left(\frac{\varphi(n)}{n}\right)=\int_0^1 F(x)\,dv(x)+O\left(e^{-\ln(2R-\varepsilon)\left(\frac{\ln\frac{N}{M}}{\ln\ln N+\ln 12(2R-\varepsilon)}-1\right)}\right).$$

Theorem 4. Let $F(x)$ be a function given on the interval $[0,1]$. Suppose that $F(x)$ has on the interval $[0,1]$ an r-th derivative ($r\ge 0$) satisfying a Lipschitz condition of first order,

$$\left|F^{(r)}(x_1)-F^{(r)}(x_2)\right|<C|x_1-x_2|.$$

Then

$$\frac{1}{N}\sum_{n\le N}F\left(\frac{\varphi(n)}{n}\right)=\int_0^1 F(x)\,dv(x)+O\left(\frac{C(\ln\ln N)^{r+1}}{\left(\ln\frac{CN}{M}\right)^{r+1}}\right).$$

2. Consider the function introduced by E. V. Novoselov ([4]):

$$\|n\|=1-\sum_{d|n}\frac{1}{2^d}.$$

Let us denote by $t(n)$

$$t(n)=\sum_{d|n}\frac{1}{2^d}.$$

It can be proved that for every real λ there exists the limit *

$$\lim_{N\to\infty}\frac{P_N(t(n)<\lambda)}{N}=v(\lambda). \tag{3}$$

In order to formulate a theorem on the rate of convergence to the limit in relation (3), it is necessary to introduce several definitions. On the half-interval $[1/2,1)$ we consider dyadic-rational points, i.e. numbers of the form

$$s = \frac{1}{2} + \frac{1}{2^{n_2}} + \frac{1}{2^{n_3}} + \dots + \frac{1}{2^{n_k}},$$

where $n_1 = 1 < n_2 < \dots < n_k$ ($n_1, n_2, \dots, n_k$ are natural numbers). Denote by $D(s)$ the least common multiple of the numbers $n_1, n_2, \dots, n_k$. A dyadic-rational point s lying on the half-interval $[1/2, 1)$ will be called a point of work if the system of exponents $1, n_2, \dots, n_k$ corresponding to this point has the following property: if $d \mid D(s)$ and $d \le n_k$, then d is among the numbers $1, n_2, \dots, n_k$. Any dyadic-rational point lying on $[1/2, 1)$ and not being a point of work will be called a point of rest. Near each dyadic-rational point

$$s = \frac{1}{2} + \frac{1}{2^{n_2}} + \frac{1}{2^{n_3}} + \dots + \frac{1}{2^{n_k}}$$

we consider, to the right, the interval $\left(s, s + \frac{1}{2^{n_k}}\right)$. If s is a point of work, then this interval will be called an interval of work; if s is a point of rest, then the interval will be called an interval of rest. We divide the points λ of the half-interval $[1/2, 1)$ into three categories.

1) Dyadic-rational points λ. It is easy to prove that every dyadic-rational point is the right endpoint of an interval of rest. Denote by $\delta(\lambda)$ the length of the longest interval of rest adjacent to λ on the left.

2) Points λ that are not dyadic-rational, but lie on some interval of rest (α, β). Denote

$$\delta(\lambda) = \max(\beta - \lambda, \lambda - \alpha).$$

3) Points λ that are not dyadic-rational and do not lie on any interval of rest. Let

$$\lambda = \frac{1}{2} + \frac{1}{2^{n_2}} + \frac{1}{2^{n_3}} + \cdots$$

be the dyadic expansion of λ. Denote, for a natural m,

$$s_m = \frac{1}{2} + \frac{1}{2^{n_2}} + \dots + \frac{1}{2^{n_m}},$$

$$\delta_m(\lambda) = \min\left(\lambda - s_m,\ s_m + \frac{1}{2^{n_m}} - \lambda\right).$$

* The theorem was communicated to me by E. V. Novoselov.

As $m \to \infty$,

$$\frac{\ln n_m}{\delta_m(\lambda)} \to \infty.$$

For a given sufficiently large natural number N, denote by $m(N)$ the greatest number m for which still

$$\frac{\ln n_m}{\delta_m(\lambda)} \leqslant 2^{\frac{\ln N}{\ln(2\ln N)}}.$$

Theorem 5. Let λ belong to the half-open interval $[1/2, 1)$.

1) If λ is a dyadic-rational number, then

$$\frac{P_N(t(n)<\lambda)}{N} = \vartheta(\lambda) + O\left(\frac{1}{\delta(\lambda)} \frac{1}{\frac{\ln \frac{N}{\delta(\lambda)}}{\ln\left(2\ln \frac{N}{\delta(\lambda)}\right)}}\right).$$

2) If λ belongs to an interval of rest, then

$$\frac{P_N(t(n)<\lambda)}{N} = \vartheta(\lambda) + O\left(\frac{1}{\delta(\lambda)} \frac{1}{\frac{\ln \frac{N}{\delta(\lambda)}}{\ln\left(2\ln \frac{N}{\delta(\lambda)}\right)}}\right).$$

3) If λ is not a dyadic-rational number and does not belong to any interval of rest, then

$$\frac{P_N(t(n)<\lambda)}{N} = \vartheta(\lambda) + O\left(\frac{1}{\ln n_{m(N)}}\right).$$

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