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Abstract

Full Text

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ON THE STATISTICS OF DEPENDENT TRIALS

(Presented by Academician A. N. Kolmogorov on 14 XI 1962)

Theorem 1. Let a random variable ξ have distribution function $F^{(i)}(x)$ at the i-th moment of time $(i \in I)^*$ and let $F_n(x)$ be the empirical distribution function of trials linked into an inhomogeneous Markov chain with nonzero ergodicity coefficients (see [6]) α_i, where

$$\min_{1\le i<n} \alpha_i = O(n^{-\beta}) \quad (0 \le \beta < 1).$$

Then, if the distribution functions $F^{(i)}(x)$ $(i \in I)$ are uniformly absolutely continuous,

$$\mathbf{P}\left\{ \sup_{-\infty<x<+\infty} \left| F_n(x) - \frac{1}{n}\sum_{i=1}^{n} F^{(i)}(x) \right| \to 0,\ n \to \infty \right\} = 1.$$

Theorem 2. Let $F(x)$ be the distribution function of a random variable ξ, and let $F_n(x)$ be the empirical distribution function of the results of n observations of the variable ξ according to the law of a homogeneous Markov chain with nonzero coefficient of ergodicity. Then

$$\mathbf{P}\left\{ \sup_{-\infty<x<+\infty} |F_n(x) - F(x)| \to 0,\ n \to \infty \right\} = 1.$$

Obviously, both of these theorems are generalizations of the well-known theorem of V. I. Glivenko [1].

Proof of Theorem 1. Since the distribution functions $F^{(i)}(x)$ $(i \in I)$ are uniformly absolutely continuous, for any $r \in I$ there exists a certain net of points with coordinates x_k^r $(x_k^r < x_{k+1}^r;\ \pm k \in I)$ on the real line R such that

$$\left| F^{(i)}(x_{k+1}^r) - F^{(i)}(x_k^r) \right| < 1/r$$

for any k $(\pm k \in I)$.

Denote the random event $\{\xi < x_k^r\}$ by A, so that the probability of its occurrence at the i-th moment of time is $P^{(i)}(A) = F^{(i)}(x_k^r)$ $(i \in I)$, and the frequency of its occurrence is $F_n(x_k^r)$. Let

$$\Phi_n(x) = F_n(x) - \frac{1}{n}\sum_{i=1}^{n} F^{(i)}(x), \qquad \delta_{n,k}^r = F_n(x_{k+1}^r) - F_n(x_k^r),$$

$$E_k^r = \{|\Phi_n(x_k^r)| \to 0,\ n \to \infty\} \quad (\pm k,\ r \in I),$$

$$E^r = \bigcap_{\pm k \in I} E_k^r = \left\{ \sup_{\pm k \in I} |\Phi_n(x_k^r)| \to 0,\ n \to \infty \right\} \quad (r \in I),$$

$$E = \bigcap_{r \in I} E^r = \left\{ \sup_{\pm k,\ r \in I} |\Phi_n(x_k^r)| \to 0,\ n \to \infty \right\},$$

$$S^r = \left\{ \overline{\lim_{n\to\infty}} \sup_{x \in R} |\Phi_n(x)| < 1/r \right\},$$

$$S = \bigcap_{r \in I} S^r = \left\{ \sup_{x \in R} |\Phi_n(x)| \to 0,\ n \to \infty \right\}.$$

* I is the set of all natural numbers.

From Theorem 3 of paper ([2]) it follows that $P(E_k^r) = 1$ $(\pm k,\ r \in I)$; if we note that Lemma 2, p. 328 of paper ([3]), is valid for arbitrarily dependent random events, we obtain that $P(E^r) = P(E) = 1$.

For the given x and r there exists one and only one k such that $x_k^r \leqslant x \leqslant x_{k+1}^r$, whence

$$F_n(x_k^r) \leqslant F_n(x) \leqslant F_n(x_{k+1}^r), \qquad F^{(i)}(x_k^r) \leqslant F^{(i)}(x) \leqslant F^{(i)}(x_{k+1}^r).$$

Consequently,

$$\Phi_n(x_k^r) - \frac{1}{r} < \Phi_n(x_{k+1}^r) - \delta_{n,k}^r \leqslant \Phi_n(x) \leqslant \Phi_n(x_k^r) + \delta_{n,k}^r \leqslant \Phi_n(x_{k+1}^r) + \frac{1}{r},$$

so that

$$|\Phi_n(x)| \leqslant \max\{|\Phi_n(x_k^r)|,\ |\Phi_n(x_{k+1}^r)|\} + \frac{1}{r} < \sup_{\pm k \in I} |\Phi_n(x_k^r)| + \frac{1}{r}$$

for any $x \in R$, i.e.

$$\sup_{x \in R} |\Phi_n(x)| \leqslant \sup_{\pm k \in I} |\Phi_n(x_k^r)| + \frac{1}{r},$$

whence $E^r \subset S^r,\ E \subset S$, and from $P(E) = 1$ it follows that $P(S) = 1$.

Let us note that this result also remains valid under the conditions of Theorem 2 of paper ([7]).

Proof of Theorem 2.
Let us note that in the proof of Glivenko' s theorem (see ([3])) the following

were used: 1) the fact that the random variable ξ has the same distribution at all instants of time; 2) Lemma 2, p. 328, of paper ([3]), which, as we have already observed, is valid for arbitrarily dependent random variables; and 3) the strengthened law of large numbers for independent trials (the Borel–Cantelli theorem). It is easy to see that whenever, for dependent trials, the strengthened law of large numbers is satisfied in this form, the proof remains valid. If, in addition, Theorem 6 from ([2]) is taken into account, then Theorem 2 is proved.

Let us note that our Theorem 1 is also valid under the conditions in which in ([4]) (Ch. II, § 12, Theorems 1 and 2; Ch. IV, § 31, Theorems 1 and 2) and in ([5]) (Ch. VIII, § 69) the strengthened law of large numbers is proved.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.