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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

**A. F. TIMAN**

## **ON THE METRIZATION THEOREM OF P. S. URYSOHN**

*(Presented by Academician A. N. Kolmogorov on 20 XI 1962)*

In this note we consider certain extremal problems of an approximation character, adjoining the well-known metrization theorem of P. S. Urysohn.

Let  $Q$  be an arbitrary regular topological space with a countable base (see (1), Ch. VI), and let  $M(Q)$  be the Banach space of all real functions  $f(x)$ , defined at points  $x \in Q$ , for which

$$\|f\| = \sup_{x \in Q} |f(x)| < \infty. \quad (1)$$

By virtue of P. S. Urysohn's theorem, the space  $Q$  is metrizable and, as is easy to see, a possible metrization of  $Q$  preserving the topology of the space is not unique. The corresponding metric space generated by the introduction in  $Q$  of some metric  $\rho(x, y)$  will be denoted by  $Q_\rho$ , and the class of functions  $f(x)$  from  $M(Q)$  satisfying the Hölder condition

$$|f(x) - f(y)| \leq \rho(x, y), \quad (2)$$

will be denoted by  $H(Q_\rho)$ .

It is natural that the transition in  $Q$  from a given metric  $\rho(x, y)$  to some other metric  $r(x, y)$  (a change of "scale" in  $Q$ ), leading, generally speaking (if one goes beyond the conditions of P. S. Urysohn's theorem), to a new topology in  $Q$ , entails a change in the class under consideration of functions  $f(x) \in M(Q)$  satisfying condition (2). In this connection there arises the question of a metric characteristic of such a change, connected with the study of the quantity which in the theory of approximation of functions is customarily called the deviation of the classes  $H(Q_r)$  and  $H(Q_\rho)$ , i.e. the quantity

$$\mathcal{E}_{H(Q_\rho)}\{H(Q_r)\} = \sup_{f \in H(Q_r)} \inf_{g \in H(Q_\rho)} \|f - g\|. \quad (3)$$

Many problems, both within the theory of approximation of functions itself and outside it, lead directly to such a formulation of the question. The following theorem gives a complete solution of this question.

**Theorem 1.** *For any metrics  $r(x, y)$  and  $\rho(x, y)$ , defined in a regular topological space  $Q$  with a countable base, the equality*

$$\mathcal{E}_{H(Q_\rho)}\{H(Q_r)\} = \frac{1}{2} \sup_{x, y \in Q} \{r(x, y) - \rho(x, y)\}, \quad (4)$$

*holds, in which the right-hand side may be either finite or infinite.*

It should be noted that the conditions for applicability of this result are in fact somewhat broader than indicated in the statement of the theorem. Equality (4) holds, for example, in all those cases when the metric spaces under consideration are separable and one of the corresponding two metrics is no weaker than the other. Simple examples show that, in this case, the topologies in the spaces  $Q_r$  and  $Q_\rho$  may turn out to be essentially different.

A number of problems encountered in the analysis of concrete problems of approximation character adjoin the general question under consideration. I shall note some of such problems and the results pertaining to them.

Denote by  $Q^n$  the unit cube of the  $n$ -dimensional real or complex space of points  $x(x_1, \dots, x_n)$ , ( $|x_k| \leq 1$ ;  $k = 1, \dots, n$ ), and, for some  $p \geq 1$ , consider the class  $H_p(Q^n)$  of all real func-

functions  $f(x_1, \dots, x_n)$  satisfying on  $Q^n$  the Hölder condition in the metric

$$\rho_p(x, y) = \left\{ \sum_{k=1}^n |x_k - y_k|^p \right\}^{1/p}, \quad (5)$$

i.e., those for which

$$|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)| \leq \rho_p(x, y). \quad (6)$$

For any  $q \geq p$  it is required to determine the magnitude of the best uniform approximation on  $Q^n$  of an arbitrary function  $f(x_1, \dots, x_n) \in H(Q^n)$  by functions  $g(x_1, \dots, x_n) \in H_q(Q^n)$ .

**Theorem 2.** If  $1 \leq p \leq q$ , then

$$\mathcal{E}_{H_q(Q^n)}\{H_p(Q^n)\} = n^{1/p} - n^{1/q}. \quad (7)$$

Let  $S$  be the unit sphere in the space  $\tilde{C}$  of all continuous  $2\pi$ -periodic functions, where  $\|x(t)\| = \max_t |x(t)|$ .

**Theorem 3.** The deviation of the class of all real-valued functionals  $f(x)$  defined on  $S$  and satisfying the condition

$$|f(x) - f(y)| \leq \|x - y\|,$$

from the class of functionals satisfying the condition

$$|f(x) - f(y)| \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |x(t) - y(t)|^p dt \right\},$$

does not depend on  $p$  and, for any finite value  $p \geq 1$ , is equal to one.

Let  $D^N$  be the set of all complex functions  $N$  times continuously differentiable on the interval  $[0, 1]$ , for which  $\max_{0 \leq t \leq 1} |x^{(k)}(t)| \leq 1$  ( $k = 0, 1, \dots, N$ ), and for any nonnegative  $m \leq N$

$$\rho_a^{(m)}(x, y) = \sum_{k=0}^m a_k \max_{0 \leq t \leq 1} |x^{(k)}(t) - y^{(k)}(t)|,$$

where  $a_k \geq 0$  ( $k = 0, 1, 2, \dots$ ) is a given sequence of nonnegative numbers. For certain problems of the calculus of variations, where the closeness of functions is considered depending on the smoothness classes to which they belong, the problem of the magnitude of the deviation, for given  $m \leq n \leq N$ , of the set  $H[D_{\rho_a}^N]$  of real-valued functionals  $f(x)$  satisfying the condition associated with the first variation

$$|f(x) - f(y)| \leq \rho_a^{(n)}(x, y),$$

from the set  $H[D_{\rho_a}^N]$  of real-valued functionals satisfying the condition

$$|f(x) - f(y)| \leq \rho_a^{(m)}(x, y)$$

may be of some interest.

**Theorem 4.** If  $m \leq n \leq N$ , then

$$\mathcal{E}_{H[D_{\rho_a}^N]} \{H[D_{\rho_a}^N]\} = \sum_{k=m+1}^n a_k. \quad (8)$$

The application of Theorem 1 to other problems in a number of cases is connected with its generalization to spaces whose metric satisfies only the conditions  $\rho(x, x) = 0$ ,  $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ . Some applications of this theorem to the

problem of the mutual deviation of classes of continuous functions of one real variable were given by me earlier in (2).

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## References

1. F. Hausdorff, *Set Theory*, Moscow-Leningrad, 1937.
2. A. F. Timan, DAN, **140**, No. 2, 307 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

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