SovietRxiv

Soviet-era science, translated into English

V. I. GAVRILOV

1963

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196301.42271

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

V. I. GAVRILOV

BOUNDARY BEHAVIOR OF FUNCTIONS MEROMORPHIC IN THE UNIT DISK

(Presented by Academician I. G. Petrovskii, January 23, 1963)

§ 1. Recently Lange ([1]), studying the distribution of values of functions holomorphic in the unit disk $D : |z| < 1$, established the existence in D, for certain classes of holomorphic functions, of sequences of non-Euclidean disks (ρ-disks in Lange' s terminology) analogous to the classical filling disks of Milloux for entire functions. More precisely, if a function $w = f(z)$, holomorphic in the disk D, has $\lim\limits_{n\to\infty} f(z_n) = \infty$ along some sequence of points $\{z_n\}$, $z_n \in D$, $|z_n| \to 1$, $n \to \infty$, and there exist a number μ_0, $0 < \mu_0 < +\infty$, and a sequence of points $\{z'_n\}$, $z'_n \in D$, at which the function $w = f(z)$ takes the value 0 or 1, while the non-Euclidean distances

$$\rho(z_n, z'_n) = \frac{1}{2} \ln \frac{1+u}{1-u}, \quad u = \left| \frac{z'_n - z_n}{1 - \bar{z}_n \cdot z'_n} \right|, \quad n = 1, 2,$$

are less than μ_0: $\rho(z_n, z'_n) < \mu_0$, $n = 1, 2, ...$, then in D there exists a sequence of nonintersecting non-Euclidean disks $\{D_n\}$,

$$D_n : \ \{z : \rho(z, \zeta_n) < 1/n\}$$

with non-Euclidean centers $\{\zeta_n\}$, $|\zeta_n| < |\zeta_{n+1}|$, $n = 1, 2, ...$, $\lim\limits_{n\to\infty} |\zeta_n| = 1$, and non-Euclidean radii $\{1/n\}$, such that in each disk D_n, $n = 1, 2, ...$, the function $w = f(z)$ assumes all values w from the disk $|w| < n$, except for a set of values w of diameter less than $2/n$. In particular, in the union

$$\bigcup_{k=1}^{\infty} D_{n_k},$$

where $\{D_{n_k}\}$ is any infinite subsequence of the disks $\{D_n\}$, the function $w = f(z)$ assumes all finite values infinitely often, with the possible exception of one. Lange called the sequences $\{D_n\}$ and $\{\zeta_n\}$, respectively, a sequence of ρ-disks and a sequence of ρ-points for the holomorphic function $w = f(z)$.

The question arises of the possibility of obtaining, to some extent, an analogous assertion for meromorphic functions. We shall say that a function $w = f(z)$,

SovietRxiv

meromorphic in the disk D, possesses in D a sequence of P-points if in D there exists a sequence of points $\{\zeta_n\}$, $|\zeta_n| < |\zeta_{n+1}|$, $n = 1, 2, ...$, $\lim_{n\to\infty} |\zeta_n| = 1$, $\lim_{n\to\infty} \rho(\zeta_n, \zeta_{n+1}) = \infty$, such that for every infinite subsequence $\{\zeta_{n_k}\}$ of it the following holds: whatever the number $\varepsilon > 0$, in the union

$$\bigcup_{k=1}^{\infty} \Delta_{n_k}^{(\varepsilon)}$$

of non-Euclidean disks

$$\Delta_{n_k}^{(\varepsilon)} : \ \{z : \rho(z; \zeta_{n_k}) < \varepsilon\}$$

with non-Euclidean centers $\{\zeta_{n_k}\}$ and non-Euclidean radii $\varepsilon > 0$, the meromorphic function $w = f(z)$ assumes all values w infinitely often, with the possible exception of two.

Theorem 1. *Suppose that there exists a sequence of positive numbers $\{\delta_n\}$, $\lim_{n\to\infty} \delta_n = 0$, and a sequence of points $\{\zeta_n\}$, $\zeta_n \in D$, $|\zeta_n| \to 1$, $n \to \infty$, along which a function $w = f(z)$, meromorphic in D, has $\lim_{n\to\infty} f(\zeta_n) = a$ (finite or infinite), while for each number n there exists, in the non-Euclidean disk with non-Euclidean center ζ_n and non-Euclidean radius δ_n, a point $\tilde{\zeta}_n$ at which $|f(\tilde{\zeta}_n) - a| > \varepsilon_0$, if the number a is finite ($|f(\tilde{\zeta}_n)| < 1/\varepsilon_0$, if the number a is infinite), where $\varepsilon_0 > 0$ is fixed. Then from the sequence $\{\zeta_n\}$ one can select a subsequence that is a sequence of P-points for the function $w = f(z)$.*

Remark 1. In ([2]) an example is constructed of a Blaschke product $w = B(z)$ with positive real zeros $\{x_n\}$, $x_n \to 1$, $n \to \infty$, such that $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 1/2$, and the function $w = B(z)$ does not have radial boundary value 0 at the point $z = 1$. This example shows that the condition $\lim_{n\to\infty} \delta_n = 0$ in Theorem 1 cannot be replaced by the condition $\lim_{n\to\infty} \delta_n = \delta_0$, $\delta_0 > 0$.

Theorem 1 is an immediate consequence of the following lemma.

Lemma 1. *If the function $w = f(z)$ satisfies the conditions of Theorem 1, then the point $z = 0$ is an irregular point (in the sense of Montel) for the sequence of functions $\{g_n(z)\}$,*

$$g_n(z) = f\left(\frac{z + \zeta_n}{1 + \bar{\zeta}_n z}\right).$$

The validity of this lemma can easily be established with the aid of the arguments from ([2]). Indeed, suppose that the family of functions $\{g_n(t)\}$,

$$g_n(t) = f\left(\frac{t+\zeta_n}{1+\bar{\zeta}_n t}\right),$$

is normal in some disk $|t| \le \lambda$, $0 < \lambda < 1$. Put

$$t_n = \frac{\tilde{\zeta}_n - \zeta_n}{1 - \bar{\zeta}_n \tilde{\zeta}_n}.$$

Then $\rho(0, t_n) = \rho(\zeta_n, \tilde{\zeta}_n)$ and, consequently, $\lim_{n\to\infty} \rho(0, t_n) = 0$. Hence $\lim_{n\to\infty} t_n = 0$. We have

$$\lim_{n\to\infty} g_n(t_n) = \lim_{n\to\infty} f(\tilde{\zeta}_n) \ne a.$$

This contradicts the equicontinuity of the family $\{g_n(t)\}$ in the disk $|t| \le \lambda$, since

$$\lim_{n\to\infty} g_n(0) = \lim_{n\to\infty} f(\zeta_n) = a.$$

The lemma is proved.

Taking into account the result of Lange stated above, one can sharpen Theorem 1 in the case when the function $w = f(z)$ is holomorphic in D.

Theorem 1′. *Let the function $w = f(z)$, holomorphic in the disk D, satisfy the conditions of Theorem 1. Then the function $w = f(z)$ has in D a sequence of ρ-points $\{z_n\}$ and*

$$\lim_{n\to\infty} \rho(z_n, \zeta_n) = 0.$$

§ 2. The following theorems concern meromorphic functions possessing sequences of P-points.

With the aid of the methods from ([3]) and Lange' s results ([1]), the following can be proved:

Theorem 2. *A sequence of points $\{\zeta_n\}$, $\zeta_n \in D$, $|\zeta_n| < |\zeta_{n+1}|$, $n = 1, 2, ...$, $\lim_{n\to\infty} \rho(\zeta_n, \zeta_{n+1}) = \infty$, is a sequence of P-points (ρ-points) for a function $w = f(z)$ meromorphic (holomorphic) in the disk D if and only if*

$$\lim_{n\to\infty} (1 - |\zeta_n|^2)\, |f'(\zeta_n)|/(1 + |f(\zeta_n)|^2) = \infty.$$

SovietRxiv

In other words, only functions meromorphic (holomorphic) in D that are not normal in D in the sense of Lehto and Virtanen possess sequences of P-points (ρ-points).

Indeed, let $\{\zeta_n\}$ be a sequence of P-points for a function $w = f(z)$ meromorphic in D. Then in any neighborhood of the point $z = 0$ every infinite subsequence $\{g_{n_k}(z)\}$ of the sequence of functions $\{g_n(z)\}$,

$$g_n(z) = f\left(\frac{z+\zeta_n}{1+\bar{\zeta}_n z}\right),$$

assumes infinitely often all values w, except possibly two. That is, the point $z = 0$ is an irregular point for the family $\{g_n(z)\}$ and, according to Marty' s criterion,

$$\lim_{n\to\infty} |g_n'(0)|/(1+|g_n(0)|^2) = +\infty$$

or

$$\lim_{n\to\infty} (1-|\zeta_n|^2)|f'(\zeta_n)|/(1+|f(\zeta_n)|^2) = +\infty.$$

Conversely, let $\{\zeta_n\}$ not be a sequence of P-points for $w = f(z)$. Then there exists $\varepsilon_0 > 0$ such that in every non-Euclidean disk

$$D_n^0 = \{z : \rho(z, \zeta_n) < \varepsilon_0\}$$

the function $w = f(z)$ does not assume three values a, b, c. According to the normality criterion for functions ([4]),

$$\frac{|f'(z)|}{1+|f(z)|^2}\,|dz| \le$$

$\le C\,d\sigma,\ z \in D_n,\ n = 1, 2, \dots$, where $d\sigma$ is the element of the hyperbolic metric in D_n, and the constant $C < +\infty$ is one and the same for all the disks $\{D_n\}$. In particular, at the points $z = \zeta_n,\ n = 1, 2, \dots$, the quantities $|dz|/d\sigma = \varepsilon_0(1-|\zeta_n|^2)$, and

$$(1-|\zeta_n|^2)\,|f'(\zeta_n)|/(1+|f(\zeta_n)|^2) \le C/\varepsilon_0 < +\infty, \quad n = 1, 2, \dots.$$

Theorems 3 and 4 are consequences of Theorem 1.

Theorem 3. *Let the function $w = f(z)$, meromorphic in the disk D, have the asymptotic value α (finite or infinite) along some Jordan curve going to*

SovietRxiv

the boundary point $\zeta_0 = e^{i\theta_0}$, and suppose that $w = f(z)$ does not have at ζ_0 the angular boundary value α. Then the function $w = f(z)$ possesses in D a sequence of P-points $\{\zeta_n\}$, $\zeta_n \to \zeta_0 = e^{i\theta_0},\ n \to \infty$.

Theorem 4. *Let the function $w = f(z) \not\equiv \mathrm{const}$, meromorphic in the disk D, have the asymptotic value α (finite or infinite) along some Jordan curve from D, the set of limit points of which on the boundary of the disk D consists of more than one point. Then the function $w = f(z)$ possesses a sequence of P-points $\{\zeta_n\}$, $|\zeta_n| \to 1,\ n \to \infty$.*

Remark 2. If, in the hypotheses of Theorems 3 and 4, the function $w = f(z)$ is assumed holomorphic, then $w = f(z)$ possesses the corresponding sequences of ρ-points.

Let, for example, the function $w = f(z)$ satisfy the hypotheses of Theorem 3. Then it satisfies the hypotheses of the Lehto–Virtanen theorem ((4), Theorem 1), and hence the following holds: in D there exists a Jordan curve Λ_0, ending at the boundary point ζ_0 and containing a sequence of points $\{z_n\}$, $z_n \to \zeta_0,\ n \to \infty$, at which $f(z_n) = a$, or b, or c, where the numbers $a, b, c \neq \alpha$, and, in addition, for any number $N > 0$ there exists in D a curve Λ_N along which $f(z) \to \alpha$ and the Euclidean distance between Λ_N and Λ_0 is less than $1/N$. (The assertion of Theorem 1 is stated in (4) in a somewhat different form, but in the course of its proof the existence of the curve Λ_0 named above is established.) Thus, if the function $w = f(z)$ satisfies the hypotheses of Theorem 3, then it also satisfies the hypotheses of Theorem 1.

Theorem 4 is proved similarly. For this purpose, in the preceding argument one must rely not on the Lehto–Virtanen theorem, but on its generalization obtained by Beagemil and Seidel ((5), Theorem 1).

Before formulating the next result, let us introduce notation. Let $S \subset D$ be a spiral, i.e. a Jordan curve given by a continuous complex-valued function $z = z(t)$, $0 < t < +\infty$, for which $0 < |z(t)| < 1$, $|z(t)| \to 1$, $\arg z(t) \to +\infty$ as $t \to +\infty$. Fix an arbitrary point $z(t) \in S$ corresponding to the value t of the parameter, and let t' be the first value of the parameter greater than t for which

$$\arg z(t') = \arg z(t) + 2\pi.$$

Consider the quantity

$$\bar{\mu}(S) = \lim_{t \to +\infty} \rho(z(t), z(t')).$$

As a consequence of Lemma 1 we have

Theorem 5. *Let the function $w = f(z)$, meromorphic in the disk D, unbounded near the unit circle, be bounded on some spiral S with $\bar{\mu}(S) = 0$. Then the function $w = f(z)$ possesses in D a sequence of P-points.*

SovietRxiv

§ 3. The result of Lange presented in § 1 admits the following refinement.

Theorem 6. *Let the function $w = f(z)$, holomorphic in the disk D, have*

$$\lim_{n\to\infty} f(z_n) = \infty$$

along some sequence of points $\{z_n\}$, $z_n \in D$, $|z_n| \to 1$, $n \to \infty$, and suppose there exist a number μ_0, $0 < \mu_0 < +\infty$, and a sequence of points $\{z'_n\}$, $z'_n \in D$, such that $\rho(z_n, z'_n) < \mu_0$, $n = 1, 2, \dots$, and the function $w = f(z)$ is bounded on $\{z'_n\}$: $|f(z'_n)| < M$, $n = 1, 2, \dots$, with a fixed constant $M < +\infty$. Then in D there exists a sequence of ρ-points for the function $w = f(z)$.

Indeed, consider the sequence of functions $\{g'_n(z)\}$,

$$g'_n(z) = f\left(\frac{z + z'_n}{1 + \overline{z'_n} z}\right)$$

and show that this sequence does not form a normal family in the disk $|z| \leq \lambda_0$, where $1 > \lambda_0 > \operatorname{th} \mu_0$. Suppose, on the contrary, that the family of functions $\{g'_n(t)\}$,

$$g'_n(t) = f\left(\frac{t + z'_n}{1 + \overline{z'_n} t}\right),$$

is normal in the disk $|t| \leq \lambda_0$. Put

$$t'_n = \frac{z_n - z'_n}{1 - \overline{z'_n} z_n};$$

then

$$\rho(0, t'_n) = \rho(z_n, z'_n) < \mu_0 < \frac{1}{2} \ln \frac{1 + \lambda_0}{1 - \lambda_0},$$

i.e. $|t'_n| < \lambda_0$. Since $|g'_n(0)| = |f(z'_n)| < M$, $n = 1, 2, \dots$, the limit function of the sequence $\{g'_n(t)\}$ must be holomorphic in $|t| < \lambda_0$, whereas

$$\lim_{n\to\infty} g'_n(t'_n) = \lim_{n\to\infty} f(z_n) = \infty.$$

Consequently, there exists a point z_0, $|z_0| < \lambda_0$, in every neighborhood of which the functions of the family $\{g'_n(z)\}$ assume, infinitely often, all finite values except possibly one. This means that there exists a sequence of points $\{\tilde{z}'_n\}$, $\rho(z_n, \tilde{z}'_n) \leq \lambda_0 + \mu_0$, $n = 1, 2, \dots$, at which $f(\tilde{z}'_n) = 0$ or 1, while $\lim_{n\to\infty} f(z_n) =$

SovietRxiv

∞. According to Lange' s result, the function $w = f(z)$ has a sequence of ρ-points in D.

If the arguments given above are slightly refined, one can establish the following fact.

Lemma 2. Let a function $w = f(z)$, holomorphic in the disk D, satisfy the conditions of Theorem 6. Then, on the segments of non-Euclidean straight lines joining pairwise the points z_n, z_n', $n = 1, 2, ...$, one can choose a sequence of points $\{\tilde{z}_n\}$ (possibly $\tilde{z}_n = z_n$, $n = 1, 2, ...$) such that an irregular point for the sequence of functions $\{\tilde{g}_n(z)\}$,

$$\tilde{g}_n(z) = f\left(\frac{z + \tilde{z}_n'}{1 + \overline{\tilde{z}_n'} z}\right),$$

is the point $z = 0$.

Hence, taking into account Lange's result cited in § 1 (1), we obtain the following refinement of Theorem 1 from (6).

Theorem 7. Let a function $w = f(z)$, holomorphic in the disk D, have

$$\lim_{n\to\infty} f(z_n) = \infty$$

for some sequence of points $\{z_n\}$, where $z_n = r_{ne}^{i\theta_0}$, $0 \leq \theta_0 \leq 2\pi$,

$$\lim_{n\to\infty} r_n = 1,$$

and let there exist a sequence of points $\{z_n'\}$, $z_n' = r_n' e^{i\theta_0}$,

$$\lim_{n\to\infty} \rho(z_n', z_n) = \mu_0 < +\infty,$$

at which the function satisfies $|f(z_n')| < M$, $n = 1, 2, ...$, where $M < +\infty$ is a constant. Then in every angle $\Delta_\delta(\theta_0) : \{z;\ |\arg(1 - e^{-i\theta_0} z)| < \delta,\ \delta > 0\}$ there is a sequence of ρ-points for the function $w = f(z)$.

We note that other assertions from (6) also admit refinement.

Received
19 I 1963

SovietRxiv

CITED LITERATURE

1. L. Lange, Ann. Sci. Ecol. Norm. Sup., (3), **77**, 257 (1960).
2. F. Bagemihl, F. Seidel, Ann. Acad. Sci. Fenn., Ser. AI, 280 (1960).
3. O. Lehto, Comm. Math. Helv., **33**, No. 3, 196 (1959).
4. O. Lehto, K. Virtanen, Acta Math., **97**, 1-2, 46 (1957).
5. F. Bagemihl, F. Seidel, Arch. d. Math., **11**, 263 (1960).
6. L. Lange, Nagoya Math. J., **19**, 41 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.