



Soviet-era science, translated into English

S. Yu. MASLOV

1963

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Abstract

Full Text

S. Yu. MASLOV

ON THE STRICT REPRESENTABILITY OF SETS BY CALCULI

(Presented by Academician P. S. Novikov on 5 IV 1963)

1. Numerous examples of specifying sets by indicating initial elements and rules for constructing new elements lead to the concept of a generated set*, refined by a number of authors with the aid of such concepts as “canonical calculus” (Post, (2)), “formal system” ((Curry, for example, (3)) and others (4–7). The identification of the common features of these clarifying concepts leads to the definitions formulated below.

By a **language of generation** (abbreviated **language**) we shall mean any collection of the form (A, Ω, Π) , where: 1) A is an arbitrary alphabet not containing the letter \square , called below the **alphabet of the language of generation**; 2) Ω is a set of algorithms over the alphabet $A \cup \{\square\}$ such that whenever an algorithm is applicable to a system of A -words**, it gives as result an A -word (we shall call these algorithms **operations of the language of generation**); 3) Π is a set of predicates over A -words; to each predicate in Π there is assigned its **arity** (number of arguments).

Let x be a letter not occurring in the description of the language of generation. Expressions of the form x_0, x_1, x_2, \dots will be called variables. If \mathcal{Y} is the designation of the language (A, Ω, Π) , then by **\mathcal{Y} -terms** we shall mean A -words, variables, and expressions of the form $\omega(T_1 \square T_2 \square \dots \square T_k)$, where ω is an operation from Ω and T_1, T_2, \dots, T_k are \mathcal{Y} -terms. Expressions of the form $(T_1, T_2, \dots, T_l \in \pi)$ ***, where π is a predicate from Π of arity l and T_1, T_2, \dots, T_l are \mathcal{Y} -terms, will be called **\mathcal{Y} -formulas**. Any expression of the form

$$T_1, T_2, \dots, T_m, F_1, F_2, \dots, F_n \vdash T_0 \quad (m \geq 0, n \geq 0), \quad (1)$$

where T_μ ($0 \leq \mu \leq m$) are \mathcal{Y} -terms, and F_ν ($0 \leq \nu \leq n$) are \mathcal{Y} -formulas, will be called a **\mathcal{Y} -scheme**. The number m will be called the **index** of the scheme (1). Schemes whose index is zero will be called **initial** or **axiom schemes**. The result of substituting certain A -words in place of all occurrences of variables**** in a \mathcal{Y} -scheme will be called a **realization** of the scheme, provided that under such a substitution all terms occurring in the scheme have meaning and all assertions expressed by the formulas of the scheme are true.

By a **basis of generation in the language \mathcal{Y}** (abbreviated **basis in \mathcal{Y}**) we shall mean any pair $(\mathcal{Y}, \mathfrak{B})$, where \mathcal{Y} is an arbitrary language of generation, called

below the **language of the basis** $(\mathcal{Y}, \mathfrak{B})$, and \mathfrak{B} is an arbitrary set of \mathcal{Y} -schemes, called below the **basis schemes** $(\mathcal{Y}, \mathfrak{B})$. If \mathfrak{B} is the set of all \mathcal{Y} -schemes, then the basis $(\mathcal{Y}, \mathfrak{B})$ will be called **complete**. If \mathfrak{B} is an enumerable set of \mathcal{Y} -schemes, then the basis $(\mathcal{Y}, \mathfrak{B})$ will be called **enumerable**. The alphabet, operations, and predicates of the language of the basis will be called, respectively, the **alphabet**,

* In this communication the term “set” will always be understood in the sense in which it is used in constructive mathematics (see, for example, (1)). We shall consider only sets of words, although everything that follows is not difficult to formulate for sets whose elements are ordered constructive objects of other types.

** An A -word is any word in the alphabet A (including the empty one); a system of A -words is any word in the alphabet $A \cup \{\square\}$.

*** $(T_1, T_2, \dots, T_l \in \pi)$ is the notation for the assertion “the list T_1, T_2, \dots, T_l satisfies the predicate π .”

**** Equal words are substituted for different occurrences of the same variable.

operations and predicates of the basis. For any generative basis Ω , any finite set of schemes of the basis Ω will be called a calculus in the generative basis Ω . In the usual way we shall understand the expressions “the word P_0 is derivable from the words P_1, P_2, \dots, P_m in one step by application of the scheme H ,” “derivation in a calculus,” “the word P is derivable in a calculus.” With the formulated definition of a calculus, already the first of the relations mentioned may turn out (for some bases) to be algorithmically unverifiable.

2. Fix some algorithmic one-to-one mapping of the natural number series onto the set of all A -words. The complete ordering of A -words generated by this mapping will be denoted by the sign $>$ (read: “greater”). A generative basis Ω will be called weakly decidable (decidable) if one can construct an algorithm which, for every scheme H of the basis Ω and every set of words $P_0, P_1, P_2, \dots, P_m$, where m is the index of the scheme H , recognizes whether the word P_0 is derivable from the words P_1, P_2, \dots, P_m (respectively, whether it is possible to construct a word P , greater than P_0 , derivable from the words P_1, P_2, \dots, P_m) in one step by application of the scheme H . From the weak decidability of Ω it follows that the assertion “the list of words Ω is a derivation in the calculus Ω in the basis Ω ” is algorithmically checkable; for any calculus in a weakly decidable basis the set of words derivable in this calculus is enumerable. If the complete basis Ω is weakly decidable, then all predicates of the basis Ω are decidable and, for any operation ω of the basis \mathfrak{B} , the relation “the word Q is the result of applying ω to the system of words $Q_1 \square Q_2 \square \dots \square Q_k$ ” is decidable. From the decidability of the complete basis \mathfrak{B} follow the weak decidability of Ω and the decidability of the domains of applicability of every operation of the basis.

As a rule, the calculi used in mathematical logic are constructed in decidable bases (exceptions are, for example, the calculi defined in (4)).

3. We shall say that a calculus Ω strictly represents a set \mathfrak{M} if \mathfrak{M} coincides with the set of words derivable in Ω . We shall say that a set \mathfrak{M} is strictly representable in the basis \mathfrak{A} if one can construct a calculus in the basis \mathfrak{A} which strictly represents \mathfrak{M} . It is known (for example (8)) that not for every enumerable set can one construct a canonical calculus which strictly represents it.* Analogous assertions are also valid for many other variants of the notion of a calculus. Facts of this kind are generalized by the following theorems.

Theorem 1. *Whatever weakly decidable enumerable generative basis \mathfrak{A} is taken, for every infinite enumerable set of A -words \mathfrak{K} (where A is the alphabet of the basis \mathfrak{A}) one can construct an infinite decidable subset of the set \mathfrak{K} which is not strictly representable in the basis \mathfrak{A} .*

We give the construction of the required set. All schemes of the basis \mathfrak{A} form an enumerable set; denote its i -th element by H_i . Choose an arbitrary A -word M_0 belonging to \mathfrak{K} , and denote the set $\{M_0\}$ by \mathfrak{M}_0 . Suppose that a finite set of A -words \mathfrak{M}_{n-1} has already been constructed; denote its greatest element by M_{n-1} . One can find a word M_n belonging to \mathfrak{K} , greater than M_{n-1} , and such that for any scheme H_i ($1 \leq i \leq n$) and any set of words belonging to \mathfrak{M}_{n-1} , in which the number of terms is equal to the index of the scheme H_i , either M_n is not derivable from this set in one step by application of H_i , or else such a word M' is derivable that

$$M_n > M' > M_{n-1}.$$

Denote by \mathfrak{M}_n the set $\mathfrak{M}_{n-1} \cup \{M_n\}$. The set \mathfrak{M} , which is the union of the sets $\mathfrak{M}_0, \mathfrak{M}_1, \mathfrak{M}_2, \dots$, will be the required one.

* By canonical calculi in an alphabet A one cannot strictly represent any set \mathfrak{M} of A -words such that

$$\forall n \exists P_1 P_2 [P_1 \in \mathfrak{M} \& P_2 \in \mathfrak{M} \& \text{dl}(P_2) \geq n \cdot \text{dl}(P_1) \& \forall P (\text{dl}(P_1) < \text{dl}(P) < \text{dl}(P_2) \supset P \notin \mathfrak{M})]$$

(n is a variable for natural numbers; P_1, P_2, P are variables for A -words; $\text{dl}(Q)$ denotes the length of the word Q).

Corollary. Whatever the weakly solvable enumerable generation basis B may be, the class of sets strictly representable in B is a proper part of the class of enumerable sets.

4. We shall say that an infinite set \mathfrak{M} **bounds the sparseness** of an infinite set \mathfrak{A} , if one can specify a word R such that

$$\forall R_1 R_2 [(R_1 \in \mathfrak{A} \& R_2 \in \mathfrak{A} \& R_1 > R_2 > R) \supset \exists K (K \in \mathfrak{M} \& R_1 > K > R_2)]. \quad (2)$$

We shall say that an infinite set \mathfrak{M} **bounds the sparseness of the sets belonging to the class Ξ** , if for every set \mathfrak{R} belonging to Ξ one can construct such a word R that either* the set \mathfrak{R} is finite, or condition (2) is satisfied.

Theorem 2. Whatever the solvable enumerable generation basis B may be, if B is weakly solvable**, then one can construct an infinite solvable set of A -words (where A is the alphabet of the basis B) which bounds the sparseness of the sets strictly representable in B .

The construction of the required set is analogous to the construction of the set \mathfrak{M} in Theorem 1. The fact used is that, under the assumptions of Theorem 2, the following is true.

Lemma. Whatever the scheme H of the basis B and the A -word M may be, one can construct a word M' such that $M' > M$, and for any collection of words P_1, P_2, \dots, P_m smaller than M (the number of words in the collection is equal to the index of the scheme H), either one can construct a word P_0 , derivable from P_1, P_2, \dots, P_m in one step of application of the scheme H , such that $M' > P_0 > M$, or no word greater than M is derivable from these words in one step of application of H .

5. If, in the formulations of Theorems 1 and 2 and of the corollary, the word “enumerable” is omitted, or if, in the formulations of Theorem 1 and the corollary, the words “weakly solvable” are omitted, or if, in the formulation of Theorem 2, the word “solvable” is omitted, then the resulting assertions may prove false. Fix an alphabet A . One can construct an enumerable (not weakly solvable) basis B_1 such that, for every enumerable set of A -words \mathfrak{M} , one can construct a calculus in the basis B_1 that strictly represents \mathfrak{M} , the only scheme of which will be the axiom scheme $(x_0 \in \mathfrak{p}) \vdash x_0$, where \mathfrak{p} is a predicate of the basis B_1 (see, for example, (4)). One can construct a solvable and weakly solvable basis B_2 such that no enumerable set of A -words can fail to be strictly representable in the basis B_2 . On the other hand, the following holds.

Theorem 3. Whatever the weakly solvable generation basis B may be, there is no algorithm which, for each enumerable set \mathfrak{M} of A -words (where A is the alphabet of the basis B), constructs a calculus in the basis B which strictly represents \mathfrak{M} .

Remark. Theorems 1–3 remain valid also for certain generalizations of the concept of a generation language, connected, for example, with the introduction of variables of different kinds, the generalization of the concept of a formula, etc.

In conclusion, the author expresses his deep gratitude to N. A. Shanin for his attention to this work.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
4 IV 1963

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* The connective “either” is understood as the double negation of disjunction.

** Instead of weak solvability of B , one may require that, for any scheme H of the basis B and any collection of words whose number of terms is equal to the index of H , the set of words derivable from this collection in one step of application of H be enumerable.

Note: Figure translations are in progress. See original paper for figures.

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