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Abstract

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ON THE NONLINEAR THEORY OF THE DEVELOPMENT OF AERODYNAMIC DISTURBANCES

At the present time, studies in the theory of the aerodynamic stability of laminar flows have become a large independent branch of aerodynamics. On the basis of the linear theory of stability, in which disturbances are assumed to be so small that their square is neglected, a number of very interesting and practically important results have been obtained. However, because of the simplifications adopted, the conclusions of the linear theory concerning the stability of a flow and the character of the development of disturbances in time will be valid only for sufficiently small time intervals (stability in the small).

For investigating the stability of an initial laminar flow over a sufficiently large time interval, it is necessary to take account of the nonlinear terms in the equations of motion. For this purpose, in the author' s work ([1]) the method of successive approximations—the usual small-parameter method—was applied. It was shown that successive approximations behave in time analogously to the first approximation, which coincides with the usual linear theory. In the unstable region the first approximation increases without bound in time, and this property is preserved (and even amplified) in the subsequent approximations ([1]). In this connection it seems expedient to use a somewhat different small-parameter method.

In the present work, for investigating the development of disturbances in time, a small-parameter method analogous to Poincaré' s method ([2]) is applied. For the successive approximations a recurrent system of ordinary differential equations is obtained. In this case the stability boundaries and the character of the development of disturbances in the successive approximations are determined, as before, by the first approximation. However, now the first approximation differs essentially from the usual linear theory. The disturbances vary in time not according to an exponential law and prove to be bounded as time increases without limit.

Consider the plane motion of a viscous incompressible fluid whose stream function satisfies the equation

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi_0}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} = \nu \nabla^2 \nabla^2 \psi. \tag{1}$$

Let the motion under consideration consist of a principal stationary motion with stream function $\psi_0(x, y)$ and a disturbed nonstationary motion with stream function $\psi_1(x, y, t)$. The stream function of the principal flow will satisfy equation (1) or the corresponding equation in boundary-layer theory. In the motion of a fluid between two parallel walls or in a boundary layer, the stream function of the disturbed flow ψ_1 will satisfy the equation:

$$\mathcal{L}(U_0 \psi_1) = \frac{\partial \nabla^2 \psi_1}{\partial t} + U_0 \frac{\partial \nabla^2 \psi_1}{\partial x} - U_0'' \frac{\partial \psi_1}{\partial x} - \nu \nabla^2 \nabla^2 \psi_1 =$$

$$= \frac{\partial \psi_1}{\partial x} \frac{\partial \nabla^2 \psi_1}{\partial y} - \frac{\partial \psi_1}{\partial y} \frac{\partial \nabla^2 \psi_1}{\partial x} \tag{2}$$

and certain boundary conditions on the surface of the body and on the outer boundary of the layer ([1]).

We write equation (2) in the following form:

$$\frac{\partial \nabla^2 \psi_1}{\partial t} + \mathcal{L}_0(U_0 \psi_1) = N(\psi_1 \psi_1). \tag{3}$$

We shall seek the solution of this equation in the form

$$\psi_1(x, y, t) = \sum_{n=1}^{\infty} \varepsilon^n A_n(\xi) \Phi_n(x, y, \xi), \tag{4}$$

where

$$t = \xi + \sum_{n=1}^{\infty} \varepsilon^n T_n(\xi). \tag{5}$$

Here ε is a small parameter, and $T_n(\xi)$ is a set of as yet unknown functions. Suppose that the series (5) and (4) converge sufficiently well for small values of ε and t. Substituting (5) and (4) into equation (3) and collecting terms with like powers of ε, we obtain the following system of differential equations:

$$\frac{dA_1}{d\xi} \nabla^2 \Phi_1 + A_1 \left\{ \frac{\partial \nabla^2 \Phi_1}{\partial \xi} + \mathcal{L}_0(U_0 \Phi_1) \right\} = 0,$$

$$\frac{dA_2}{d\xi}\nabla^2\Phi_2 + A_2\left\{\frac{\partial\nabla^2\Phi_2}{\partial\xi} + \mathcal{L}_0(U\Phi_2)\right\} = A_1^2 N(\Phi_1\Phi_1) - A_1\frac{dT_1}{d\xi}\mathcal{L}_0(U_0\Phi_1), \quad (6)$$

$$\frac{dA_3}{d\xi}\nabla^2\Phi_3 + A_3\left\{\frac{\partial\nabla^2\Phi_3}{\partial\xi} + \mathcal{L}_0(U_0\Phi_3)\right\} =$$

$$= A_1A_2\widetilde{N}(\Phi_1\Phi_2) - A_2\mathcal{L}_0(U_0\Phi_2)\frac{dT_1}{d\xi} + A_1^2\frac{dT_1}{d\xi}N(\Phi_1\Phi_1) - A_1\frac{dT_2}{d\xi}\mathcal{L}_0(U_0\Phi_1),$$

. . . .

and, in general, for $n > 2$,

$$\frac{dA_n}{d\xi}\nabla^2\Phi_n + A_n\left\{\frac{\partial\nabla^2\Phi_n}{\partial\xi} + \mathcal{L}_0(U_0\Phi_n)\right\} = \sum_{k=2}^{n-1} T'_{n-k}\sum_{s=1}^{k-1} A_{sA_{k-s}}N(\Phi_s\Phi_{k-s}) +$$

$$+\sum_{k=1}^{n-1} A_k\left\{A_{n-k}N(\Phi_k\Phi_{n-k}) - \frac{dT_{n-k}}{d\xi}\mathcal{L}_0(U_0\Phi_k)\right\}.$$

As is seen, this system of differential equations will not depend on the coefficients $A_n(\xi)$, if the following conditions are satisfied:

$$dA_n/d\xi = \gamma_1 n A_n(\xi), \quad (7)$$

$$\frac{dT_n}{d\xi} = A_n(\xi), \quad (8)$$

where γ_1 is a certain constant. Using expressions (7) and (8), we reduce system (6) to the form:

$$\mathcal{L}_{\gamma_1}(U_0\Phi_1) = 0,$$

$$\mathcal{L}_{2\gamma_1}(U_0\Phi_2) = N(\Phi_1\Phi_1) + \gamma_1\nabla^2\Phi_1 + \frac{\partial\nabla^2\Phi_1}{\partial\xi},$$

$$\mathcal{L}_{3\gamma_1}(U_0\Phi_3) = \widetilde{N}(\Phi_1\Phi_2) + 2\gamma_1\nabla^2\Phi_2 + \frac{\partial\nabla^2\Phi_2}{\partial\xi}, \quad (9)$$

. . . .

$$\mathcal{L}_{n\gamma_1}(U_0\Phi_n) = \sum_{k=1}^{n-1} \frac{\partial \Phi_k}{\partial x}\frac{\partial \nabla^2\Phi_{n-k}}{\partial y} - \frac{\partial \Phi_k}{\partial y}\frac{\partial \nabla^2\Phi_{n-k}}{\partial x} + (n-1)\gamma_1\nabla^2\Phi_{n-1} + \frac{\partial \nabla^2\Phi_{n-1}}{\partial \xi}.$$

Here the notation has been adopted

$$\mathcal{L}_{n\gamma_1}(U_0\Phi_n) = n\gamma_1\nabla^2\Phi_n + \frac{\partial \nabla^2\Phi_n}{\partial \xi} + \mathcal{L}_0(U_0\Phi_n). \tag{10}$$

The first equation of this recurrent system of differential equations is equivalent to the basic equation of linear stability theory. Therefore the coefficient $\gamma_1(\alpha, \mathrm{Re})$ will be the known coefficient of linear stability theory characterizing the law of variation of disturbances in time. The system of differential equations (9) is analogous to the correspond-

of a system of differential equations for periodic functions $F_n(x, y, t)$, obtained by the author in [1]. To solve this system, one may apply the method indicated there of reducing system (9) to a system of ordinary differential equations and to quadratures.

We shall find the dependence of the coefficients A_n on the variable ξ from expression (7)

$$A_n(\xi) = \lambda^n e^{\gamma_1 \xi n} = \{A_1(\xi)\}^n. \tag{11}$$

As can be seen, the character of the dependence of the coefficients on the auxiliary variable ξ is analogous to the character of the dependence of the corresponding coefficients in the nonlinear theory of stability on time (the variable t).

Using expressions (8) and (11), for values $\gamma_1 \neq 0$ we find

$$T_n = \lambda^n e^{\gamma_1 n \xi}/\nu_1 n + c_n. \tag{12}$$

Substituting these expressions into (5), we obtain

$$\gamma_1 t = \gamma_1 \xi + \sum_{n=1}^{\infty} \varepsilon^n \frac{\lambda^n}{n} e^{\gamma_1 n \xi} + c.$$

For values $\lambda \varepsilon e^{\gamma_1 \xi} = \varepsilon A_1 < 1$, this series can be summed:

$$\gamma_1 t = \gamma_1 \xi - \ln(1 - \varepsilon A_1) + \ln(1 - \varepsilon\lambda). \tag{13}$$

Eliminating the auxiliary variable ξ from expressions (12) and (13), we find the dependence of the amplitude A_1 on time t in the form:

$$A_1(t) = \frac{\lambda e^{\gamma_1 t}}{1 - \varepsilon\lambda + \lambda\varepsilon e^{\gamma_1 t}}. \tag{14}$$

Using this expression, from (13) we obtain the dependence $\xi(t)$

$$\gamma_1 \xi = \gamma_1 t - \ln(1 - \varepsilon\lambda + \lambda\varepsilon e^{\gamma_1 t}). \tag{15}$$

Setting the auxiliary parameter ε in the expressions given above equal to unity, we finally obtain the following expressions for the stream function of the disturbed flow:

$$\psi_1(x, y, t) = \sum_{n=1}^{\infty} A_1^{(n)}(t)\Phi_n(x, y, \xi(t)), \tag{16}$$

$$A_1(t) = \frac{\lambda e^{\gamma_1 t}}{1 - \lambda + \lambda e^{\gamma_1 t}}, \tag{17}$$

$$\gamma_1 \xi = \gamma_1 t - \ln(1 - \lambda + \lambda e^{\gamma_1 t}), \tag{18}$$

where the quantity $\gamma_1(a, \mathrm{Re})$ is determined by the linear theory of stability.

Let us note first of all that in the first approximation we shall have

$$\psi_1^{(1)}(x, y, t) = \frac{\lambda e^{\gamma_1 t}}{1 - \lambda + \lambda e^{\gamma_1 t}} e^{i(\alpha x - \omega\xi(t))}.$$

Therefore, in the stable region of the flow, $\gamma_1(a, \mathrm{Re}) < 0$, the first approximation for sufficiently large t will tend to zero, while in the unstable region of the flow, $\gamma_1 > 0$, it will stabilize in time. As can be seen, the subsequent approximations will behave in time analogously to the first approximation. Accordingly, from (16)–(18), for the stable region of the flow $\gamma_1 < 0$ and for $t \gg 0$ we obtain an expression for the stream function

$$\psi_1(x, y, t) = \sum_{n=1}^{\infty} \lambda^n e^{\gamma_1 n t}\Phi_n(x, y, t). \tag{19}$$

As can be seen, as $t \to \infty$ the stream function tends to zero, and the initial laminar flow proves stable with respect to disturbances $\psi_1(x, y, 0)$.

In the unstable region of the flow, for $\gamma_1(a, \mathrm{Re}) > 0$ and for sufficiently large t, from expressions (16)–(18) we shall formally have

$$A(\xi) \approx 1, \qquad \gamma_1 \xi \approx \text{const}, \qquad \psi_1(x, y, t) \approx \sum_{n=1}^{\infty} \Phi_n(x, y, \text{const}). \tag{20}$$

As is seen, from the convergence conditions of the series (20) there will follow stabilization of the disturbances in time. As $t \to \infty$, the disturbed flow will tend to some new, likewise laminar flow, in the present case independent of time. Consequently, under the indicated conditions, the initial laminar flow U_0 in the range of values $\gamma_1 > 0$ will be unstable and will pass into a new, likewise laminar flow $u = U_0 + \partial\psi_1/\partial y,\ v = -\partial\psi_1/\partial x$.

Let us note that the process of successive approximations can be constructed differently, which will correspond to other initial conditions. We shall now seek a particular solution of equations (3) in the form:

$$\psi_1(x, y, t) = \sum_{n=1}^{\infty} \varepsilon_n A_n(\xi) \psi_n(x, y, t), \tag{21}$$

where ε and ξ have the same meanings as in expressions (4) and (5). Suppose also that this series converges sufficiently well for small values of ε and t. Then, carrying out entirely analogous calculations, we obtain the following expression for the stream function:

$$\psi_1(x, y, t) = \sum_{n=1}^{\infty} A_1^n(t) \psi_n(x, y, t), \tag{22}$$

where $A_1(t) = \lambda e^{\gamma_1 t}/(1 - \lambda + \lambda e^{\gamma_1 t})$.

As is seen, in the stable region of the flow, $\gamma_1 < 0$, the stream function will as before tend to zero. In the unstable region of the flow, for $\gamma_1 > 0$ and for sufficiently large t, from expressions (22) we shall have

$$\psi_1(x, y, t) \approx \sum_{n=1}^{\infty} \psi_n(x, y, t). \tag{23}$$

Consequently, in the present case, from the convergence conditions of the series (23) there will also follow stabilization of the disturbances in time. However, now the disturbed flow as $t \to \infty$ will tend to a new laminar flow periodic in time.

The application of the modified small-parameter method has led to a substantial change in the first and subsequent approximations. In the first—linear—approximation the disturbances will decay for $\gamma_1 < 0$ and stabilize in time for $\gamma_1 > 0$.

The subsequent approximations will behave in time analogously to the new first approximation. Therefore, when the convergence conditions of the method of successive approximations are satisfied, the disturbed flow as a whole will also behave analogously to the first approximation, namely:

For $\gamma_1 < 0$ the disturbances will decay in time and the initial laminar flow will be stable. For $\gamma_1 > 0$ the disturbances will stabilize in time, giving rise to a new, likewise laminar flow.

L. D. Landau first proposed the hypothesis of the possibility of transition of an initial laminar flow into a new, likewise laminar flow ([3]). There are a number of works in the literature in which the solutions of the linear problem were modified in such a way that they led to stabilization in time. Following Landau, such expressions were obtained by a number of authors, but they satisfy neither the approximate nor the exact equations of hydrodynamics.

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Note: Figure translations are in progress. See original paper for figures.