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Abstract

Full Text

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ON THE THEORY OF SEMIGROUPS OF LINEAR OPERATORS

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1°. Denote by $\widetilde{C}$ the space of all continuous 2π-periodic functions $f(x)$ with norm

$$\|f\| = \max_{0 \le x < 2\pi} |f(x)|.$$

Let U_n be a linear operator from $\widetilde{C}$ into $\widetilde{C}$, taking every $f \in \widetilde{C}$ into a trigonometric polynomial of order $\le n$. Introduce the operator

$$\widetilde{U}_n(f) = \frac{1}{2\pi} \int_0^{2\pi} (U_n(f_t))_{-t} \, dt,$$

where $f_t(x) = f(x+t)$. In the paper ([1]) it was proved that for any $f \in \widetilde{C}$ the equality

$$\widetilde{U}_n(f, x) = \int_0^{2\pi} f(x+t)\Phi_n(t) \, dt, \tag{1}$$

holds, where $\Phi_n(t) = \widetilde{U}_n(D_n, -t)$ and D_n is the Dirichlet kernel of order n. Hence it follows that

$$\|U_n\| \ge \int_0^{2\pi} \left|\widetilde{U}_n(D_n, t)\right| dt. \tag{2}$$

We shall now abandon the requirement that, for every $f \in \widetilde{C}$, $U_n(f)$ be a polynomial of order n, and consider an arbitrary semigroup

$$\Omega = \{U_\xi, \ \xi > 0\}$$

of linear bounded operators, measurable with respect to ξ, and mapping the space $\widetilde{C}$ into itself.

It turns out that for the operators Ω there are analogues of equality (1) and inequality (2).

2°. Define the operator

$$\widetilde{U}_\xi(f) = \frac{1}{2\pi} \int_0^{2\pi} (U_\xi(f_t))_{-t} \, dt. \tag{3}$$

Theorem 1. Let Ω be an arbitrary semigroup of linear bounded operators from $\widetilde{C}$ into $\widetilde{C}$, measurable with respect to ξ. Then the operators

$$\Omega_1 = \{\widetilde{U}_\xi,\ \xi > 0\},$$

where the operator $\widetilde{U}_\xi$ is defined by formula (3), also form a semigroup of linear bounded operators measurable with respect to ξ; moreover, the operators Ω_1 have the following properties:

1) The operators Ω_1 map $\widetilde{C}$ into $\widetilde{C}$.
2) The operators Ω_1, being constructed for arbitrary operators Ω, commute with the group of real translations, i.e.

$$\widetilde{U}_\xi(f_t) = (\widetilde{U}_\xi(f))_t, \qquad f \in \widetilde{C}, \qquad -\infty < t < \infty.$$

3) If the operators Ω commute with the group of real translations, then $\Omega_1 \equiv \Omega$.
4) $\|\widetilde{U}_\xi\| \leqslant \|U_\xi\|$.

Proof. Property 1) is obvious.

Let us prove property 2). To this end we compute $(\widetilde{U}_\xi(f_t))_{-t}$. According to equality (3) we have

$$(\widetilde{U}_\xi(f_t))_{-t} = \frac{1}{2\pi}\int_0^{2\pi} (U_\xi(f_{t+t_1}))_{-(t+t_1)}\, dt_1. \tag{4}$$

Since $(U_\xi(f_t))_{-t}$ is a 2π-periodic function of t, it follows from equality (4) that

$$(\widetilde{U}_\xi(f_t))_{-t} = \frac{1}{2\pi}\int_0^{2\pi} (U_\xi(f_z))_{-z}\, dz.$$

Consequently, $(\widetilde{U}_\xi(f_t))_{-t} = \widetilde{U}_\xi(f)$, or $\widetilde{U}_\xi(f_t) = (\widetilde{U}_\xi(f))_t$. Thus property 2) is proved.

Property 3) is obvious.

Let us prove property 4). For any $-\infty < t < \infty$, $(U_\xi(f_t))_{-t} \in \widetilde{C}$. Therefore the integral on the right-hand side of equality (3) may be regarded as a Bochner integral ([2]). It is known ([2]) that for the Bochner integral the inequality

$$\left\|\int f\, dt\right\| \leqslant \int \|f\|\, dt$$

holds. Hence it follows from equality (3) that

$$\|\widetilde{U}_\xi(f)\| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \|(U_\xi(f_t))_{-t}\|\,dt. \tag{5}$$

Since U_ξ is a linear operator and $\|f_t\| = \|f\|$, it follows from inequality (5) that $\|\widetilde{U}_\xi(f)\| \leqslant \|U_\xi\|\|f\|$.

Theorem 2. Let $U_\xi \in \Omega_1$. Then there exist a set N of integers and a set $\Lambda = \{\lambda_j\}_{j\in N}$ of complex numbers such that the function*

$$K(t,\xi) = \sum_{j\in N} \frac{e^{jti}-1}{2\pi ji}\, e^{\lambda_j \xi} \tag{6}$$

is of bounded variation on $[-\pi,\pi]$ for every $\xi > 0$, and for every $f \in \widetilde{C}$ the equality

$$\widetilde{U}_\xi(f,x) = \int_{-\pi}^{\pi} f(x-t)\, d_{tK}(t,\xi) \tag{7}$$

holds.

This theorem follows directly from Theorem 1 and the theorem of Hille ([2]), according to which, for a semigroup of operators from $\widetilde{C}$ into $\widetilde{C}$ commuting with the group of real translations, representation (7) holds.

Theorem 3. Let $U_\xi \in \Omega$. Then the inequality

$$\|U_\xi\| \geqslant \operatorname*{Var}_{t\in[-\pi,\pi]} K(t,\xi), \tag{8}$$

holds, where $K(t,\xi)$ is determined from Theorem 2.

Proof. According to Theorem 1,

$$\|\widetilde{U}_\xi\| \leqslant \|U_\xi\|. \tag{9}$$

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* If $j = 0 \in N$, the corresponding term on the right-hand side of (6) is taken to be equal to t.

On the other hand, it is known that the norm of the operator

$$\sigma(f,x) = \int_{-\pi}^{\pi} f(x-t)\, d_{tK}(t,\xi),$$

which maps $\widetilde{C}$ into $\widetilde{C}$, is equal to $\operatorname{Var}_t K$. Therefore, according to Theorem 2,

$$\|\widetilde{U}_\xi\| = \mathrm{Var}_{t\in[-\pi,\pi]} K(t,\xi). \tag{10}$$

From (9) and (10), (8) follows.

Corollary. Let the operators Ω be such that $\overline{\lim}_{\xi\to\infty} \mathrm{Var}\, K(x,\xi) = \infty$. Then the relation

$$U_\xi(f,x) \to f(x), \qquad \xi \to \infty$$

cannot hold uniformly for all $f \in \widetilde{C}$.

Theorems 1-3 also remain valid for operators Ω that map $\widetilde{L}$ into $\widetilde{L}$, where $\widetilde{L}$ is the space of all summable 2π-periodic functions with norm

$$\|f\| = \int_0^{2\pi} |f(t)|\, dt.$$

In this case the proofs do not differ essentially from the proofs for the case of the space $\widetilde{C}$.

3°. Denote by L the space of all functions $f(x)$ summable on the entire real axis with norm

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

Let U be a bounded linear operator from L into L. Put

$$\widetilde{U}(f) = \lim_{\substack{\tau_1\to-\infty\\ \tau_2\to+\infty}} \frac{1}{\tau_2-\tau_1} \int_{\tau_1}^{\tau_2} (U(f_t))_{-t}\, dt. \tag{11}$$

It is obvious that $\widetilde{U}$ is also a bounded linear operator from L into L.

Theorem 4. The operator $\widetilde{U}$ has the properties:

1) For any $f \in L$ and any t, $\widetilde{U}(f_t) = (\widetilde{U}(f))_t$.

2) $\|\widetilde{U}\| \le \|U\|$.

Theorem 5. Let $\widetilde{U}$ be an arbitrary bounded linear operator from L into L. Then there exists a function $\Phi(t)$ of bounded variation on $(-\infty,\infty)$ such that, for any $f \in L$, the equality

$$\widetilde{U}(f,x) = \int_{-\infty}^{\infty} f(x-t)\, d\Phi(t)$$

holds, where the operator $\widetilde{U}$ is defined according to formula (11).

Theorem 6. Let U be an arbitrary bounded linear operator from L into L. Then

$$\|U\| \geq \int_{-\infty}^{\infty} |d\Phi(t)|,$$

where $\Phi(t)$ is defined from Theorem 5.

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CITED LITERATURE

1. D. L. Berman, DAN, **144**, No. 3 (1962).

2. E. Hille, R. Phillips, *Functional Analysis and Semigroups*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.