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Abstract

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THE SIMPLEST SOLVABLE EXTENSIONS FOR ULTRAHYPERBOLIC AND PSEUDOPARABOLIC OPERATORS

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In the work ([1]) it was established that a solvable extension ([2]) exists for an arbitrary differential operator L with constant coefficients, considered in a bounded domain Q of Euclidean space with sufficiently regular boundary. In other words, there always exists such an extension $\widetilde{L}$ of the operator L, initially considered on functions of class C^∞ that vanish on the boundary of Q, that the equation $\widetilde{L}u = f$ has in H (the Hilbert space of square-summable functions) a unique solution u for every $f \in H$. This extension must be characterized by some system of boundary conditions.

All well-posed boundary-value problems for the classical equations of mathematical physics and their generalizations determine solvable extensions of the corresponding operators. At the same time, the question of the nature of solvable extensions for "pathological" operators (for example, for the ultrahyperbolic operator) remained open. The purpose of the present article is to describe the simplest solvable extensions in a parallelepiped for certain typical "pathological" operators. The considerations are based on the use of the Fourier method in its simplest form.

A distinctive feature of the extensions presented is the specification of boundary conditions in the form

$$\gamma_1[u(s_1)] + \gamma_2[u(s_2)] = 0, \tag{1}$$

where γ_1, γ_2 are certain linear operators, and s_1, s_2 are distinct points of the boundary (the simplest example is a periodicity condition). For classical operators, boundary conditions can always be given in the form $\gamma_1[u(s_1)] = 0$. There are grounds to suspect that the use of "nonlocal" conditions of the form (1), at least in one of the variables, is unavoidable in describing solvable extensions for certain classes of operators. As the example given below shows, in a number of cases the extensions found are "unstable."

Within the framework of the methods used, it is in principle possible to study (analogously to what is done here) in a parallelepiped an arbitrary operator with constant coefficients. However, the ways of passing to domains of a more general form or to the case of variable coefficients appear unclear.

Let us pass to the main content of the work. Consider the intervals
$I = [0 \leq t \leq 1]$; $I_x = [0 \leq x_\mu \leq 2\pi]$; $J_y = [0 \leq y_\tau \leq 2\pi]$, and let $V = I_1 \times \cdots \times I_m \times J_1 \times \cdots \times J_n$; $Q = I \times V$.

In Q we shall consider the set P^∞ of complex infinitely differentiable functions of $m+n+1$ variables, periodic with period 2π in the variables x, y. For elements of P^∞ we define in the usual way the scalar product and norm:

$$(u, v) = \int_Q u\bar{v}\, dQ; \qquad |u, H|^2 = (u, u).$$

The completion of P^∞ in the introduced norm gives the Hilbert space H of square-summable functions. On the elements of P^∞ are defined diff-

differential operators

$$T_1 \equiv \frac{\partial}{\partial t} + L_x - L_y; \qquad T_2 \equiv \frac{\partial^2}{\partial t^2} + L_x - L_y,$$

where

$$L_x \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2}; \qquad L_y \equiv \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}.$$

The operator T_1 for $m \geqslant 1,\ n \geqslant 1$ will be called pseudoparabolic; for $m \geqslant 1,\ n \geqslant 2$ the operator T_2 will be called ultrahyperbolic.

Let u_0, u_1 denote the values of the function $u \in P^\infty$ on the upper and lower bases of Q, i.e. $u_\sigma = u|_{t=\sigma}$; $\sigma = 0, 1$, and introduce boundary conditions with respect to t:

$$u_0 + \lambda u_1 = 0 \tag{Γ_1}$$

or

$$u_0 = 0; \qquad \frac{\partial u_0}{\partial t} + \lambda u_1 = 0. \tag{Γ_2}$$

In what follows we shall speak of the operator T and the boundary conditions (Γ), understanding by T either of the operators T_1, T_2, and associating with the operator T_1 the conditions (Γ_1), and with the operator T_2 the conditions (Γ_2). The values of the function u and of the parameter λ will henceforth be assumed real.

We shall call a function $u \in H$ a strong solution of the equation

$$Tu = f \tag{T}$$

under the conditions (Γ) if there exists a sequence $\{u_i\}$ of functions from P^∞, satisfying the conditions (Γ), for which

$$|u - u_i, H| \to 0, \qquad |Tu_i - f, H| \to 0 \quad \text{as } i \to \infty.$$

The corresponding extension of the operator T will be called the strong extension.

Lemma 1. *For strong solutions of equation (T) under the conditions (Γ) and for all λ (except for a countable set of exceptional values) the inequality*

$$|u, H| \leqslant C|Tu, H|, \qquad (\;)$$

holds, where the constant C depends only on λ.

We outline the proof. According to the definition of a strong solution, it is sufficient to establish (Φ) for any $u \in P^\infty$ satisfying the conditions (Γ). Represent $u \in P^\infty$ in the form

$$u = \sum a_k(t) e^{ikz},$$

where $z = (x_1, \dots, x_m, y_1, \dots, y_n)$, $k = (k_1, k_2, \dots, k_{m+n})$, and the summation is over all integer values k_i, $-\infty < k_i < \infty$. Then

$$|u, H|^2 = \int_Q |u|^2 \, dQ = (2\pi)^{m+n} \sum \|a_k\|_t^2, \tag{2}$$

where we have used the notation

$$\int_0^1 |v(t)|^2 \, dt = \|v\|_t^2.$$

Obviously, for $u \in P^\infty$, $Tu \in P^\infty$ and the following equalities hold:

$$T_1 u = \sum \left(\frac{da_k}{dt} - \alpha_k^2 a_k \right) e^{ikz}; \qquad T_2 u = \sum \left(\frac{d^2 a_k}{dt^2} - \alpha_k^2 a_k \right) e^{ikz},$$

where

$$\alpha_k^2 \equiv \alpha^2(k) = k_1^2 + \dots + k_m^2 - k_{m+1}^2 - \dots - k_{m+n}^2. \tag{3}$$

Now let $u \in P^\infty$ be additionally subject to the conditions (Γ) and satisfy equation (T), with f, in turn, represented in the form

$$f = \sum \varphi_k(t) e^{ikz}.$$

Then, according to (2), to prove the inequality (Φ) it is enough to establish that, for any k,

$$\|a_k\|_t \leqslant C \|\varphi_k\|_t, \tag{4}$$

where C does not depend on k, and

$$\frac{da_k}{dt} - \alpha_k^2 a_k = \varphi_k; \qquad a_k(0) + \lambda a_k(1) = 0 \tag{A_1}$$

or

$$\frac{d^2 a_k}{dt^2} - \alpha_k^2 a_k = \varphi_k; \qquad a_k(0) = 0; \qquad \frac{da_k(0)}{dt} + \lambda a_k(1) = 0. \tag{A_2}$$

In what follows we shall systematically use the following obvious estimate: for $0 \le a < b \le 1$, $p \ne 0$, and a function $f \in H$ given on $[0,1]$,

$$\left| \int_a^b e^{p\tau} f(\tau)\, d\tau \right|^2 \le \frac{e^{2pb} - e^{2pa}}{2p} \int_a^b |f|^2\, d\tau \le \frac{M}{2p} \|f\|_t^2, \tag{5}$$

where $M = e^{2pb}$ for $p > 0$ and $M = -e^{-2pa}$ for $p < 0$.

The proof of inequality (4) for T_1, T_2 is carried out separately.

Case T_1. We shall call special the values λ given by the equalities $\lambda = -\exp[-\alpha^2(k)]$, $\lambda = 0$. For any nonspecial value λ, the solution of problem (A_1) is given by the formula

$$a_k(t) = e^{\alpha^2 t} \left[\int_0^t e^{-\alpha^2 \tau} \varphi_k\, d\tau - \frac{\lambda}{1 + \lambda e^{\alpha^2}} \int_0^1 e^{-\alpha^2 \tau} \varphi_k\, d\tau \right], \tag{6}$$

where $\alpha^2 = \alpha^2(k)$. We shall regard λ as having a fixed nonspecial value. Then there exist numbers N and M such that for any k

$$|e^{-\alpha^2} + \lambda| \ge N^{-1}; \qquad |1 + \lambda e^{\alpha^2}| \ge M^{-1}. \tag{7}$$

For $\alpha^2(k) = 0$ the validity of estimate (4) is obvious. Let now

$$1 \le \alpha^2(k) < \infty. \tag{8}$$

We represent the first of the integrals in (6) in the form $\int_0^t = \int_0^1 - \int_t^1$ and use the fact that

$$1 - \frac{\lambda e^{\alpha^2}}{1 + \lambda e^{\alpha^2}} = \frac{e^{-\alpha^2}}{e^{-\alpha^2} + \lambda}.$$

Then, using (5) and the first of inequalities (7), we obtain

$$\left| e^{\alpha^2 t} \frac{e^{-\alpha^2}}{e^{-\alpha^2} + \lambda} \int_0^1 e^{-\alpha^2 \tau} \varphi_k \, d\tau \right|^2 \le \frac{N^2}{2\alpha^2} \|\varphi_k\|_t^2.$$

The remaining term, on the basis of (5), is estimated simply by $\frac{1}{2}\alpha^{-2}\|\varphi_k\|_t^2$. As a result we obtain

$$|a_k(t)|^2 \le \frac{1}{\alpha^2}(1 + N^2)\|\varphi_k\|_t^2, \tag{9}$$

whence (4) follows for the values k corresponding to (8). In the case $-\infty < \alpha^2(k) \le -1$, an estimate analogous to (9) is obtained at once by using (5) and the second of inequalities (7). For the operator T_1, inequality (Φ) has been established.

Case T_2. We shall call special the values λ given by the equalities

$$\lambda = -2\alpha(k)\{\exp \alpha(k) - \exp[-\alpha(k)]\}^{-1}, \qquad \lambda = 0, -1.$$

For $\alpha^2(k) \ne 0$ and nonspecial λ, the solution of problem (A_2) is given by the formula

$$a_k(t) = F_k(t, \alpha) + F_k(t, -\alpha),$$

where

$$F_k(t, \alpha) = \frac{e^{\alpha t}}{2\alpha} \left\{ \int_0^t e^{-\alpha\tau} \varphi_k \, d\tau + \frac{\lambda}{2\alpha + \lambda(e^{\alpha} - e^{-\alpha})} \left[e^{-\alpha} \int_0^1 e^{\alpha\tau} \varphi_k \, d\tau - e^{\alpha} \int_0^1 e^{-\alpha\tau} \varphi_k \, d\tau \right] \right\} \tag{10}$$

and $\alpha = \alpha(k)$. For $\alpha^2 = 0$, an explicit formula for the solution of problem (A_2) gives

the presence of inequality (4) obvious. To obtain (4) from (10), it is evidently enough to establish the inequality

$$|F_k(t, \alpha)| \leq C \|\varphi_k\|_t \tag{11}$$

in the cases: $-\infty < \alpha \leq -1$; $1 \leq \alpha < \infty$; $-\infty < i\alpha \leq -1$; $1 \leq i\alpha < \infty$. In each of these cases inequality (11) can be established by elementary methods analogous to those used in the case T_1. We shall regard Lemma 1 as proved.

Corollary. The strong solution of equation (T) is unique.

Lemma 2. *For every $f \in H$, a strong solution of equation (T) exists.*

The proof follows immediately from the possibility of approximating f in H by finite sums of the form $\sum \varphi_k(t) e^{ikz}$ with smooth functions $\varphi_k(t)$.

Lemma 1 and its corollary, together with Lemma 2, can be combined into the theorem:

Theorem. *The strong extension of the operator T, corresponding to a nonspecial value λ, defines a solvable extension.*

The simplest example of instability of a solvable extension. Let in the operator T_1 $m = n = 1$ and let $\lambda = \lambda_0 < 0$ be a fixed nonspecial value of λ. If in the operator T_1 one replaces $\frac{\partial}{\partial x}$ by $\vartheta \frac{\partial}{\partial x}$, or if in the definition of V the interval $0 \leq x \leq 2\pi$ is replaced by $0 \leq x \leq 2\vartheta\pi$, where ϑ is some constant, $0 < \vartheta \leq 1$, then the expression for α_k^2 in (3) takes the form $\alpha^2 = \vartheta k_1^2 - k_2^2$. The value λ_0 will be special when $-\lambda_0 = \exp(-\alpha^2)$, or $\vartheta = k_1^{-2} \left[k_2^2 - \ln(-\lambda_0) \right]$. Since $\vartheta \to 1$ as $k_1 = k_2 = N \to \infty$, there exist ϑ' s arbitrarily close to 1 for which $\lambda = \lambda_0$ is a special value.

In the example considered, an evidently very substantial role is played by the indefiniteness of the expression for $\alpha^2(k)$ and by the choice of λ_0 negative. The half-axis $\lambda > 0$ is, from the point of view considered here, a sector of stability.

As is not difficult to verify, for the operator T_2 a sector of stability will be the interval $0 < \lambda < \alpha_0 (\sin \alpha_0)^{-1}$, where α_0 is the root of the equation $\alpha = \tan \alpha$ lying in the interval $\pi < \alpha < {}^3/_2\pi$. The sectors of stability are nothing other than intervals of regularity in the case where V is the whole space and the Fourier series is replaced by the Fourier integral.

The character of the strong extension of the operator T for special λ. Among the special values λ, one should distinguish isolated and essentially special ones: limiting points of isolated special values ($\lambda = 0$, $\pm\infty$). For an isolated special λ, solvability up to certain subspaces $\tilde{H}$, generally speaking infinite-dimensional, occurs for equation (T). For essentially special λ (corresponding to the transition to classical problems), there may occur (depending on the signature of the expression for α^2) either unique solvability or specific phenomena of ill-posedness of the Hadamard–Petrovskii type, or instability of the type occurring in the Dirichlet problem for the equation of a string.

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Note: Figure translations are in progress. See original paper for figures.