AN ALTERNATIVE PRINCIPLE FOR THE EXISTENCE OF PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH DELAYING ARGUMENT

1963

SovietRxiv

Abstract

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MATHEMATICS

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AN ALTERNATIVE PRINCIPLE FOR THE EXISTENCE OF PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH DELAYING ARGUMENT

(Presented by Academician A. Yu. Ishlinskii on April 6, 1963)

Let R^m denote m-dimensional Euclidean space, let $\tilde{x} = x(t)$ $(-\infty < t < \infty)$ be a vector function with values in R^m, and let $W(t, \tilde{x})$ be an operator with values in R^m, defined for each t on some set D of functions $\tilde{x}$. In what follows it is assumed that D contains piecewise-continuous ω-periodic vector functions.

We shall call the operator $W(t, \tilde{x})$ ω-periodic if, for every ω-periodic function $\tilde{x}$, the equality $W(t + \omega, \tilde{x}) = W(t, \tilde{x})$ holds.

Below we study the equation

$$\frac{dx}{dt} = W(t, \tilde{x}) \tag{1}$$

with an ω-periodic right-hand side. A new, apparently, method is proposed for proving theorems on the existence of periodic solutions.

Equations of type (1) include ordinary systems of differential equations

$$\frac{dx}{dt} = f(t, x) \tag{2}$$

and systems with delaying arguments (see (5, 6))

$$\frac{dx}{dt} = f[t, x(t), x(t - h_1), \dots, x(t - h_k)], \tag{3}$$

if the right-hand sides are, in the usual sense, ω-periodic in t. Numerous classes of integro-differential equations can also be written in the form (1). For example, the right-hand side of the equation

$$\frac{dx}{dt} = \varphi(t) + \int_{-\infty}^{t} K[t, s, x(s)]\, ds \tag{4}$$

will satisfy the condition of ω-periodicity if $\varphi(t+\omega) \equiv \varphi(t)$ and $K(t+\omega, s+\omega, x) \equiv K(t,s,x)$.

1. Let C denote the space of vector functions $x(t)$, continuous on $[0,\omega]$, with values in R^m. By the symbol $\tilde{x}_\omega = x_\omega(t)$ below we denote the ω-periodic continuation, from the interval $[0,\omega]$, of the function $x(t)$.

Introduce for consideration the operator

$$Ax(t) = x(\omega) + \int_0^t W(s, \tilde{x}_\omega)\, ds. \tag{5}$$

We shall assume that it acts and is completely continuous in the space C (for concrete equations of types (2)–(4), the conditions of complete continuity are obvious).

It is easy to see that the periodic solutions of equation (1) determine the fixed points of the operator (5). The converse assertion is also true: if the functions $W(t,\tilde{x})$ are piecewise continuous for piecewise continuous $\tilde{x}$, and continuous for continuous ω-periodic $\tilde{x}$, then the fixed points $x(t)$ of the operator (5) determine ω-periodic solutions $x_\omega(t)$ of equation (1). This makes it possible, in proving existence theorems for periodic solutions, to apply general methods of nonlinear functional analysis. In particular, the results of the paper [1] make it possible in a number of cases to compute the rotation on certain surfaces of the completely continuous vector field $x(t) - Ax(t)$; from this follows the assertion of the main theorem of the paper (Theorem 2).

2. Consider the system (2) of ordinary differential equations. Suppose here that the right-hand sides have such properties that each initial condition $x(0) = x_0$ uniquely determines a solution $x(t) = x(t, x_0)$ of system (2), extended to the interval $0 \le t \le \omega$. The operator $Ux_0 = x(\omega, x_0)$ is called the translation operator along the trajectories of the differential equation. The continuity of the translation operator is evident. The fixed points of the translation operator determine ω-periodic solutions of system (2). Numerous works are devoted to the study and applications of the translation operator in problems on periodic solutions.

Suppose that on the boundary Π of some domain G the translation operator has no fixed points. Denote by γ_0 the rotation on Π of the vector field $x - Ux$. If $\gamma_0 \neq 0$, then, as is known, in G there is at least one fixed point of the translation operator. Thus, the inequality $\gamma_0 \neq 0$ is a criterion for the existence of a periodic solution.

We shall say that system (2) is nondegenerate at infinity if its ω-periodic solutions are uniformly bounded and if, on spheres S of large radius, the rotation $\gamma(S)$ of the field $x - Ux$ is nonzero (this rotation, obviously, does not depend on the sphere).

The simplest criteria for nondegeneracy of the system consist in the existence of a function $V(x)$ such that

$$(\operatorname{grad} V(x), f(t,x)) > 0 \quad (|x| \geq r_0); \tag{6}$$

$$\lim_{|x|\to\infty} |V(x)| = \infty. \tag{7}$$

More general conditions for nondegeneracy of the system are given by the theorems presented in [2] (unfortunately, the proofs of the assertions from [2] concerning almost periodic solutions turned out to be incomplete; the assertions about bounded and about periodic solutions are true without changes).

Nondegeneracy of the system implies, in particular, the existence of periodic solutions. In the theorem stated below, conditions are indicated for the existence of periodic solutions. The conditions of this theorem are at the same time a criterion for nondegeneracy of the system if its solutions can be extended to the interval $0 \leq t \leq \omega$.

A function $V(x)$ for which condition (6) is fulfilled is called a guiding function for system (2); at the same time it is not assumed that condition (7) is fulfilled (the function $V(x)$ may change sign).

Theorem 1. *Suppose that for system (2) one can indicate l guiding functions $V_1(x), \dots, V_l(x)$ such that*

$$\lim_{|x|\to\infty} \{|V_1(x)| + \cdots + |V_l(x)|\} = \infty. \tag{8}$$

Suppose that the rotation of the vector field $\operatorname{grad} V_1(x)$ *on spheres of large radius is nonzero (for example, $V_1(x)$ is definite). Then system (2) has at least one solution uniformly bounded on $(-\infty, \infty)$. If the right-hand side of the system is ω-periodic, then there exists an ω-periodic solution.*

This assertion follows from the main lemma of the work [2].

3. Let us return to the study of equation (1) with a right-hand side depending on a scalar parameter λ, $0 \leqslant \lambda \leqslant 1$. As above, we shall assume that the right-hand side of this equation

$$\frac{dx}{dt} = W(t, \tilde{x}; \lambda) \tag{9}$$

is ω-periodic in t. We shall say that the right-hand side is integrally continuous with respect to the parameter λ, if the operator

$$Bx(t) = x(\omega) + \int_0^t W(s, \tilde{x}_\omega; \lambda)\, ds \tag{10}$$

is completely continuous in C, and if it depends continuously on λ, uniformly with respect to the elements of each ball of the space C. In the case of ordinary systems (2), or systems (3) of equations with retarded argument, integral continuity will obviously hold if the right-hand sides of the systems are continuous in λ. Below we assume that the right-hand side of system (9) is integrally continuous in λ.

Theorem 2*. *Suppose that for $\lambda = 0$ equation (9) becomes equation (1), and for $\lambda = 1$ it becomes system (2), which is nondegenerate at infinity. Then the existence of an a priori estimate for the ω-periodic solutions of system (9) implies the existence of at least one ω-periodic solution of system (1).*

This assertion is based on the general theory of completely continuous vector fields ([3,4]) and substantially relies on Theorem 2 from ([1]).

4. We give a simple consequence of Theorem 2.

Theorem 3. *Suppose that the ω-periodic right-hand side of system (3) with retarded argument satisfies the conditions*

$$(\operatorname{grad} V_i(x),\ f(t, x, y_1, \dots, y_k)) > 0 \quad (\|x\| \geqslant r_0;\ i = 1, \dots, l), \tag{11}$$

where the functions $V_i(x)$ satisfy condition (8). Suppose that the rotation of the field $\operatorname{grad} V_1(x)$ on the spheres $|x| = r$ of large radii r is different from zero (for example, $V_1(x)$ is even). Then system (3) has at least one ω-periodic solution.

If the right-hand side of system (3) does not possess the property of ω-periodicity, then (11) implies only the existence of a solution bounded on $(-\infty, \infty)$.

5. Denote by $\gamma_0(\Pi)$ the rotation on the boundary Π of a bounded domain $G \subset R^m$ of the vector field $x - Ux$, where U is the shift operator along the trajectories of system (2).

Consider, together with system (2), the perturbed system

$$\frac{dx}{dt} = f(t, x) + W(t, \tilde{x}; \varepsilon), \tag{12}$$

where the perturbation $W(t, \tilde{x}; \varepsilon)$ tends integrally to zero as $\varepsilon \to 0$, in the sense that

$$\lim_{\varepsilon \to 0} \left\{ \sup_{t \in [0, \omega],\, \|x\|_C \leqslant r} \left| \int_0^t W(s, \tilde{x}_\omega; \varepsilon)\, ds \right| \right\} = 0. \tag{13}$$

Integral convergence to zero is, in particular, ensured if $W(t, \tilde{x}; \varepsilon) = \varepsilon W(t, \tilde{x})$.

Theorem 4. *Suppose $\gamma_0(\Pi) \neq 0$. Then, for sufficiently small ε, system (12) has at least one ω-periodic solution.*

It follows from this assertion, in particular, that an isolated singular point of nonzero index of an autonomous system of ordinary differen-

* A related assertion was obtained independently by Yu. G. Borisovich by the method of Poincaré–Andronov transformations.

differential equations, in whose neighborhood there are no cycles, under a small perturbation by the ω-periodic operator $W(t, \tilde{x})$, "passes over" into an ω-periodic solution.

6. As a last example (of a more special, but more concrete, character), consider the system

$$\frac{dx}{dt} = Ax + W(t, \tilde{x}), \tag{14}$$

where A is a matrix whose eigenvalues have nonzero real parts. Suppose that the ω-periodic operator $W(t, \tilde{x})$ satisfies the inequality

$$\left| \int_0^t W(s, \tilde{x}_\omega)\, ds \right| \leq \alpha(\|x(t)\|_C), \tag{15}$$

and

$$\lim_{r \to \infty} \frac{\alpha(r)}{r} = 0. \tag{16}$$

Theorem 5. *Under the conditions listed, system (14) has at least one ω-periodic solution.*

Analogous theorems for ordinary differential equations are well known (B. P. Demidovich, A. Halanay, and others).

7. The method applied in the present article makes it possible, for equations (1) or (9), to prove by ordinary techniques theorems on the existence of secondary solutions (if one solution is known), to study bifurcations of periodic solutions, etc.

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Received
23 III 1963

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Note: Figure translations are in progress. See original paper for figures.