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Abstract

Full Text

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ON PROBLEMS OF BEST APPROXIMATION IN RANDOM NORMED SPACES

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The present work is devoted to problems of best approximation in random normed spaces ([1]). Let us recall the corresponding definition. Let $\mathfrak{B}$ be the set of all nonincreasing functions $\xi(x)$, continuous from the left and defined on $R = (-\infty, \infty)$, such that $\xi(x) = 1$ if $x \leqslant 0$, and $\xi(\infty) = 0$. Let B be some subset of the set $\mathfrak{B}$. For a given function α, defined on $\mathfrak{B} \times \mathfrak{B}$ with values in $\mathfrak{B}$, form the set

$$B[\alpha] = \bigcup_{n=0}^{\infty} B_n[\alpha],$$

where

$$B_0[\alpha] = B, \qquad B_n[\alpha] = B_{n-1}[\alpha] \cup \alpha(B_{n-1}[\alpha] \times B_{n-1}[\alpha]).$$

In the set $\mathfrak{B}$ introduce an order relation, putting $\xi \leqslant \eta$ if $\xi(x) \leqslant \eta(x)$ for every $x \in R$. We shall write $\xi < \eta$ if $\xi \leqslant \eta$ and there exists an $x \in R$ such that $\xi(x) < \eta(x)$. The function α will be called a B-function if for all $\xi, \eta, \zeta \in B[\alpha]$: 1) $\alpha(\xi, \eta) = \alpha(\eta, \xi)$; 2) $\alpha(\Delta, \xi) = \xi$, where Δ is the function from $\mathfrak{B}$ such that $\Delta(x) = 0$ for $x > 0$; 3) $\alpha(\xi, \eta) \geqslant \alpha(\xi_1, \eta_1)$, if $\xi \geqslant \xi_1, \eta \geqslant \eta_1$; 4) $\alpha(\alpha(\xi, \eta), \zeta) = \alpha(\xi, \alpha(\eta, \zeta))$; 5)

$$\alpha(\xi, \eta \mid x) \leqslant \inf_{t \in [0,1]} \min\{\xi(tx) + \eta((1-t)x), 1\}$$

(here $\alpha(\xi, \eta \mid x)$ is the value of the function $\alpha(\xi, \eta)$ at the point $x \in R$).

Definition. A **random normed space** (r.n.s.) is a triple (Ω, f, μ), where Ω is some linear space over the field Λ of complex or real numbers; f is a mapping of Ω into $\mathfrak{B}$:

$$\varphi \in \Omega \to f(\varphi) = \|\varphi\| = \|\varphi; \cdot\| \in \mathfrak{B};$$

μ is some $f(\Omega)$-function. At the same time the following axioms are satisfied:

I. $\|\varphi\| = \Delta$ if and only if $\varphi = \theta$ (θ is the zero element of Ω).

II. $\|a\varphi; x\| = \left\|\varphi; \dfrac{x}{|a|}\right\|$ for every $x \in R$ and every $a \in \Lambda$.

III. $\|\varphi + \psi\| \leqslant \mu(\|\varphi\|, \|\psi\|)$, $\varphi \in \Omega$; $\psi \in \Omega$.

We shall call the function f a **random norm**. In what follows we shall speak of an r.n.s. (Ω, f) or Ω, allowing such abbreviations when the remaining elements of the triple are not important.

Let $(\varphi(t),\, t \in [0, \infty))$ be some process with continuous trajectories, the observation of which ceases at a random time τ. Let $\varphi_0(t)$ be some realization of this process. It is required to approximate the function $\varphi_0(t)$ by means of a linear combination of functions from a given system $\{\varphi_1(t), \dots, \varphi_n(t)\}$ from the space $C_{[0,\infty)}$ of continuous functions on the half-line. The meaning of approximation may be understood in different ways. For example, one may seek constants $c_1, \dots, c_n$, $c_k \in R$, from the condition that for each $x > 0$ the probability be minimal that

$$\max_{0 \leqslant t \leqslant \tau} \left| \varphi_0(t) - \sum_{k=1}^{n} c_k \varphi_k(t) \right| \geqslant x.$$

According to ([2]), in the space $C_{[0,\infty)}$ one can introduce a random norm f_τ, putting

$$\|\varphi; x\| = P\left(\max_{0 \leqslant t \leqslant \tau} |\varphi(t)| \geqslant x \right).$$

Consequently, the problem just formulated is the problem of minimizing the random norm

$$\left\| \varphi_0 - \sum_{k=1}^{n} c_k \varphi_k \right\|$$

in the space $(C_{[0,\infty)}, f_\tau)$. This problem can be formulated in the general case.

Let (Ω, f) be some r.n.s., $\varphi_0 \in \Omega$, and let $\{\varphi_1, \dots, \varphi_n\}$ be a linearly independent system from Ω. We shall say that the polynomial

$$\sum_{k=1}^{n} c_k^0 \varphi_k$$

realizes the f-best approximation for φ_0, if there is no polynomial $\sum_{k=1}^{n} c_k' \varphi_k$ for which

$$\left\| \varphi_0 - \sum_{k=1}^{n} c_k' \varphi_k \right\| < \left\| \varphi_0 - \sum_{k=1}^{n} c_k^0 \varphi_k \right\|.$$

The problem of finding f-best approximations will be called the problem of f-best approximation.

We indicate one more variant of the approximation problem. We shall call the variance of the norm of an element φ of an r.n.s. the quantity

$$\sigma^2(\|\varphi\|) = 2\int_0^\infty z\|\varphi; z\|\, dz + \left[\int_0^\infty \|\varphi; z\|\, dz \right]^2,$$

provided the corresponding integrals exist. Let $\varphi_0 \in \Omega$, and let $\{\varphi_1, \dots, \varphi_n\} \subset \Omega$ be a linearly independent system. Suppose that for every $\varphi \in \{\varphi_0, \varphi_1, \dots, \varphi_n\}$ the integral

$$\int_0^\infty z\|\varphi; z\|\, dz$$

exists. It is required, among all random norms of the form

$$\left\|\varphi_0 - \sum_{k=1}^{n} c_k\varphi_k\right\|,$$

to find those that minimize the quantity

$$\sigma^2\left(\left\|\varphi_0 - \sum_{k=1}^{n} c_k\varphi_k\right\|\right).$$

We shall call this problem the problem of σ^2-best approximation. Polynomials realizing the σ^2-best approximation will be called polynomials of σ^2-best approximation.

Theorem 1. *Every polynomial of σ^2-best approximation is a polynomial of some f-best approximation.*

Let us note the connection between the problems posed and problems of best approximation in particular spaces.

Theorem 2. *Let N be a normed space with norm $p(\cdot)$. Then the problem of best approximation with respect to the norm in this space is equivalent to the problem of f-best (σ^2-best) approximation in the space N as an r.n.s. with random norm f, given in the form*

$$\|\varphi; x\| = \begin{cases} 1, & x \le p(\varphi), \\ 0, & x > p(\varphi). \end{cases}$$

In [1] it was shown that every countably normed space can be considered as an r.n.s. It is of interest to determine to what problem for a countably normed space the problem of best approximation in an r.n.s. reduces. Let (Φ, q) be an arbitrary countably normed space. Here

$$q = q(\cdot) = \{|\cdot|_1, |\cdot|_2, \ldots\}$$

is a multinorm given on Φ. On the set $\{q(\varphi)\}$ of values of q on Φ we introduce an order relation, taking $q(\varphi) \le q(\psi)$ if and only if $|\varphi|_k \le |\psi|_k$, $k = 1, 2, \ldots$. Correspondingly, $q(\varphi) < q(\psi)$ if $q(\varphi) \le q(\psi)$ and there exists a natural k for which $|\varphi|_k < |\psi|_k$. Let $\varphi_0 \in \Phi$ and let $\{\varphi_1, \ldots, \varphi_n\} \subset \Phi$ be some linearly independent system. We shall say that the polynomial

$$\sum_{k=1}^{n} c_k^0\varphi_k$$

realizes the q-best approximation for

element φ_0 among all polynomials of the form $\sum_{k=1}^{n} c_k\varphi_k$, if there is no polynomial $\sum_{k=1}^{n} c_k'\varphi_k$ for which

$$q\left(\varphi_0 - \sum_{k=1}^{n} c_k'\varphi_k\right) < q\left(\varphi_0 - \sum_{k=1}^{n} c_k^0\varphi_k\right).$$

Now one may formulate the following result.

Theorem 3. *Let (Φ, q) be a countably normed space and let the multinorm q be such that $|\cdot|_1 \le |\cdot|_2 \le \cdots$. Let τ be an arbitrary random variable taking nonnegative integer values, with $P(\tau = m) > 0,\ m = 1, 2, \ldots$. Then the problem of q-best approximation for the space (Φ, q) is equivalent to the problem of f-best approximation in the r.n.s. (Φ, f_τ) with the random norm f_τ, given by the formula $\|\varphi; x\| = P(|\varphi|_\tau \ge x)$.*

Next we shall consider questions of existence and uniqueness for problems of best approximation in an r.n.s.

Theorem 4. *Let (Ω, f) be an arbitrary r.n.s. and let $\{\varphi_1, \ldots, \varphi_n\} \subset \Omega$ be an arbitrary linearly independent system. Then, whatever $\varphi_0 \in \Phi$ may be, there always exists some polynomial $\sum_{k=1}^{n} c_k^0 \varphi_k$ which is a polynomial of f-best approximation for the element φ_0.*

We note that the proof of the theorem is based on the axiom of choice.

Theorem 5. *Let (Ω, f) be an arbitrary r.n.s., $\varphi_0 \in \Omega$, and let $\{\varphi_1, \ldots, \varphi_n\} \subset \Omega$ be an arbitrary linearly independent system. Suppose the random norm f is such that*

$$\int_0^\infty z \,\|\varphi; z\|\, dz < \infty$$

for any $\varphi \in \{\varphi_0, \varphi_1, \ldots, \varphi_n\}$. Then there always exists some polynomial $\sum_{k=1}^{n} c_k^0 \varphi_k$ which realizes the σ^2-best approximation to φ_0 by means of the system $\{\varphi_1, \ldots, \varphi_n\}$.

Theorems 4 and 5 solve the existence problem for problems of best approximation. Another important aspect is the problem of uniqueness. We shall dwell on the question of uniqueness for the problem of f-best approximation. We note that in solving this problem one has to "minimize" a quantity whose values are not numbers but functions (random norms). It may turn out that, as f-best approximations, there are several incomparable random norms, and if there are no additional requirements on the choice of a solution, we have no grounds for preferring any one solution. It may also happen that to each f-best approximation there corresponds only one polynomial realizing it. Conversely, the case is possible when the f-best approximation is unique, but there exist several realizations of this approximation. Thus, in the present case the concept of uniqueness has two aspects. We shall say that the **problem of f-best approximation has an f-unique solution** if to each f-best approximation there corresponds only one polynomial realizing this approximation. We shall say that the **problem of f-best approximation has a unique solution** if it has an f-unique solution and there exists only one f-best approximation. We note that the typical notion here is precisely that of f-uniqueness. Accordingly, we shall call a system $\{\varphi_1, \ldots, \varphi_n\} \subset \Omega$ **f-Chebyshev** if, for any $\varphi_0 \in \Omega$, the problem of f-best approximation of φ_0 by means of the system $\{\varphi_1, \ldots, \varphi_n\}$ has an f-unique solution.

Theorem 6. Let (Ω, f, μ_0) be a random normed space, where

$$\mu_0(\xi, \eta \mid x) = \inf_{t \in [0,1]} \max\{\xi(tx), \eta((1-t)x)\}.$$

If Ω is such that $\|\varphi + \psi\| = \mu_0(\|\varphi\|, \|\psi\|)$ only when $\varphi = \lambda\psi$ $(\lambda \geqslant 0)$, then every linearly independent system in Ω is f-Chebyshev.

Theorem 7. In order that the system $\{\varphi_1(t), \dots, \varphi_n(t)\}$ be f-Chebyshev in the space $(C_{[0,\infty)}, f_\tau, \mu_0)$, where τ is an arbitrary random variable taking non-negative values with strictly decreasing function $\xi(x) = P(\tau \geqslant x)$, $x > 0$, it is sufficient that every nontrivial polynomial

$$\sum_{k=1}^{n} c_k \varphi_k(t)$$

have no more than $n-1$ zeros on the half-line $[0, \infty)$.

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.