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Abstract

Full Text

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ON SIEGEL ZEROS

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In the paper (^{1}) Brauer proved the asymptotic law:

$$\ln hR \sim \ln \sqrt{|d|}, \qquad |d| \to \infty, \tag{1}$$

for any number field K of degree n over the field of rational numbers. From the analytic point of view this means that the real Siegel zero γ of the zeta-function of the field K satisfies the condition

$$1 - \gamma > \frac{c(\varepsilon)}{|d|^{\varepsilon}}, \tag{2}$$

where $\varepsilon > 0$ is an arbitrarily small constant.

In this note it will be shown how one can supplement Brauer' s idea with some considerations following from the functional equation for $\zeta_k(s)$ and Heilbronn' s theorem, and obtain a strengthening of law (1) in the form

$$\ln hR = \ln \sqrt{|d|} + O(\ln \ln |d|) \tag{3}$$

for almost all fields.

By the words "for almost all" we shall mean the following. Denote by $A_n(N)$ the number of fields of degree n whose discriminants in absolute value do not exceed N; then the number of those fields among $A_n(N)$ for which (1) holds, but possibly (3) does not hold, does not exceed the quantity

$$\exp\left[(\ln A_n(N))^{1/n}\right], \qquad N \to \infty. \tag{4}$$

First of all we note that the following generalization of Heilbronn' s theorem is valid for the Dedekind zeta-functions of a field K:

Theorem 1. If $F(s) = A^s \Gamma^{r_1}\left(\frac{s}{2}\right) \Gamma^{r_2}(s) \zeta_k(s)$, $A = \sqrt{|d|}/(\sqrt{\pi})^n 2^{r_2}$, then

$$F(s) - \frac{\operatorname{res}_{s=1} F}{s(s-1)} = \sum_{n=1}^{\infty} a_n \int_{NxNy \geq 1} \cdots \int \left[(Nx)^{\frac{1}{2}s-1}(Ny)^{s-1} + (Nx)^{\frac{1}{2}(1-s)-1}(Ny)^{(1-s)-1}\right] \times$$

$$\times \exp\left[-\left(\frac{n}{A}\right)^{1/r_1} s(x) - \left(\frac{n}{A}\right)^{1/r_2} s(y)\right] dx_1 \dots dx_{r_1} dy_1 \dots dy_{r_2},$$

$$Nx = x_1 x_2 \cdots x_{r_1}, \qquad s(x) = x_1 + x_2 + \cdots + x_{r_1},$$

$$Ny = y_1 y_2 \cdots y_{r_2}, \qquad s(y) = y_1 + y_2 + \cdots + y_{r_2},$$

a_n is the number of solutions of the equation $n = N(\mathfrak{A})$, where $\mathfrak{A}$ is an integral ideal.

The proof of this relation is carried out in complete analogy with Theorem 48 in (2).

Consider two normal fields K_1 and K_2 with discriminants d_1 and d_2, under the condition that $|d_1| \leq |d_2|$. The case of arbitrary fields is obtained from this according to the scheme proposed by Brauer in the paper (1). Let K_3 be the compositum of the fields K_1 and K_2: $K_3 = K_1 \cdot K_2$.

The zeta-function of the field K_3 will be divisible both by the zeta-function of the field K_1 and by the zeta-function of the field K_2:

$$\zeta_{K_3}(s) = \zeta_{K_1}(s) \cdot M_1(s),$$

$$\zeta_{K_3}(s) = \zeta_{K_2}(s) \cdot M_2(s).$$

Suppose that $\zeta_{K_1}(s)$ and $\zeta_{K_2}(s)$ have real Siegel zeros γ_1 and γ_2. Let us apply Theorem 1 to $\zeta_{K_3}(s)$ according to the scheme set out in detail in (2). We obtain the inequality

$$\frac{1-\gamma_2}{1-\gamma_1} > \frac{a(n)}{(\ln|d_2|)^{c_0(n)}} \exp(-(1-\gamma_1)\ln|d_2|). \tag{5}$$

From it one immediately obtains Brauer' s asymptotic law (1). Indeed, if for γ_1 we take the first Siegel zero with the condition $1-\gamma_1 < \varepsilon$, then we obtain (2) and, consequently, (1), as follows from (3). But from inequality (5) one can obtain a strengthening of law (1) in the form (3), as was said above.

For this, take as γ_1 an arbitrary Siegel zero (if such exists at all) satisfying the condition

$$1-\gamma < \frac{1}{(\ln|d_1|)^{c(n)}}, \tag{6}$$

where $c(n)$ is some constant depending only on the degree n of the field and subject to the condition $c(n) \geq 2n$. Then for all discriminants d_2 lying in the interval

$$\exp\left(\frac{1}{(1-\gamma_1)^{\frac{1}{n+1}}}\right) \leq |d_2| \leq \exp\left(\frac{1}{1-\gamma_1}\right) \tag{7}$$

we obtain from (5) the estimate of the Siegel zero:

$$1-\gamma_2 \geq \frac{a(n)}{(\ln|d_2|)^{c_0(n)+2+n_0}}.$$

Put $c(n) = \max[2n, c_0(n)+n+2]$; then in the interval (7) we have

$$1-\gamma_2 \geq \frac{1}{(\ln|d_2|)^{c(n)}}. \tag{8}$$

But the estimate (8), as follows from (^{3}), is equivalent to the asymptotic law:

$$\ln hR = \ln\sqrt{|d|} + O(\ln\ln|d|).$$

Thus this law can pass into the weaker law (1) only in the interval

$$|d_1| \leq |d_2| < \exp\left(\frac{1}{(1-\gamma_1)^{\frac{1}{n+1}}}\right). \tag{9}$$

But the number of fields of degree n whose discriminants, in absolute value, lie in the interval (9), compared with the number of fields of degree n whose discriminants lie in the interval

$$|d_1| \leq |d_2| \leq \exp\left(\frac{1}{1-\gamma_1}\right), \tag{10}$$

is a quantity of order (4), as was said at the beginning, if by N one understands the quantity

$$N = \exp\left(\frac{1}{1-\gamma_1}\right).$$

Indeed, there are no more than $|d|^{c_1(n)}$ fields of degree n having one and the same discriminant in absolute value $|d|$, where $c_1(n)$ is an absolute constant depending only on n. Consequently, the number of fields assigned to the interval (9) is no greater than the quantity

$$\exp\left(\frac{c_1(n)+1}{(1-\gamma_1)^{\frac{1}{n+1}}}\right). \tag{11}$$

But the number of all fields assigned to the interval (10) is a quantity of order not less than

$$\exp\left(\frac{1}{n(1-\gamma_1)}\right). \tag{12}$$

Indeed, a field of degree n, generated by the equation $x^n = q$, has discriminant, in absolute value, equal to $a(n)q^{n-1}$, where $a(n) > 0$ is an absolute constant depending only on n. Consequently, letting q run through the natural numbers, we obtain the theorem that discriminants of fields of degree n occur no less frequently than the $(n-1)$-st powers of integers. From this we obtain the estimate (12).

From comparison of the estimates (11) and (12) we indeed obtain the assertion that, if there are fields whose zeta-functions have Siegel zeros satisfying condition (6), then, in relation to all fields, they have order no greater than (4).

Let us make one more remark concerning the growth of the discriminants of fields whose zeta-functions have Siegel zeros satisfying condition (6). It is known that if there is a real zero which is an exception to the boundary of all zeros, then it is necessarily unique and of first order for the given $\zeta_K(s)$. From this remark and inequality (5) one may conclude that a new real zero, different from γ_1 and satisfying condition (6), can occur only for $\zeta_{K_2}(s)$ with discriminant d_2 of the field K_2, subject to the condition

$$|d_2| > \exp\left(\frac{1}{1-\gamma_1}\right). \tag{13}$$

From this inequality and inequality (6) an interesting consequence follows. Denote by d_ν the discriminant of the field K_ν, whose zeta-function has a real Siegel zero that is the ν-th in order (distinct zeros are meant) and satisfies condition (6). Then from (13) and (6) we obtain the recurrence relation:

$$|d_\nu| > \exp\left\{(\ln|d_{\nu-1}|)^{c(n)}\right\} \geq [(\ln|d_{\nu-1}|)^{2n}];$$

whence it is already easy to obtain the estimate

$$|d_\nu| > n^{n^\nu}, \qquad \nu > \nu_0.$$

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References

1. R. Brauer, *Am. J. Math.*, **69** (69), No. 2, 243 (1947).
2. N. G. Chudakov, *Introduction to the Theory of Dirichlet L-Functions*, 1947.
3. A. I. Vinogradov, *DAN*, **146**, No. 2, 274 (1962).