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ON THE STABILITY OF CHAPMAN–JOUGUET DETONATION

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Abstract

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HYDROMECHANICS

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ON THE STABILITY OF CHAPMAN–JOUGUET DETONATION

(Presented by Academician M. A. Lavrent' ev, 3 VIII 1962)

The following mathematical model of detonation is considered. In the direction of the z-axis, in the region $z < 0$, an ideal perfect gas flows with a constant supersonic velocity. In a neighborhood of the plane $z = 0$ there is a strong discontinuity. It is followed by a combustion zone, in which the equation of chemical kinetics is satisfied ([1])

$$\frac{d\beta}{dt} = -L\beta^m p^{m-1} e^{-A\rho/\mu p}.$$

Here β is the mass concentration of unreacted molecules; p is the pressure; ρ is the density; A is the activation energy; m is the order of the reaction; μ is the mean molecular weight of the mixture; L denotes certain positive constants, with $m \geqslant 1$. The chemical reaction ends when the value $\beta = 0$ is reached.

A detonation is called a Chapman–Jouguet detonation if, at $\beta = 0$, the flow velocity is equal to the local speed of sound. The point at which this equality is reached is called the Jouguet point. In this model there exists a stationary one-dimensional solution of the equations of hydrodynamics and chemical kinetics. It is described by the formulas:

$$w = w_*(1 - cx), \qquad p = p_*(1 + \gamma cx), \qquad \rho = \rho_*(1 - cx)^{-1}, \qquad \beta = x^2. \quad (1)$$

Here w is the flow velocity; $\gamma > 1$ is the adiabatic exponent; M is the Mach number in the region $z < 0$; $c = (M^2 - 1)(\gamma M^2 + 1)^{-1}$. Values of the quantities at the Jouguet point are marked with an asterisk. The function $x = x(z)$ is defined by the relation

$$\int_x^1 y^{-2m+1}(1 - cy)(1 + \gamma cy)^{-m+1} \exp\left[a(1 + \gamma cy)^{-1}(1 - cy)^{-1}\right] dy = \sigma z,$$

where

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$$a = A\mu^{-1} p_*^{-1} \rho_*, \qquad \sigma = \tfrac{1}{2} w_*^{-1} p_*^{m-1} L.$$

In this work the stability of the basic solution (1) of the equations of hydrodynamics and chemical kinetics with respect to small perturbations is investigated. An analogous question was considered in works ([2–4]). However, in them there was no rigorous mathematical formulation of the problem, and only the qualitative side of the question was clarified. In a rigorous and even somewhat more general mathematical formulation the problem was considered in work ([5]), where, however, no concrete results are reported.

In what follows we assume that the gas flows in a circular cylindrical tube of radius r_0. The flow is considered in the cylindrical coordinate system (r, φ, z) with the z-axis along the axis of the tube. Assuming that small perturbations of the flow, concentrated in the region $z > 0$, are a superposition of cylindrical harmonics, we shall study the behavior of an individual harmonic. In this case the equation of the perturbed discontinuity surface has the form

$$Z = \varepsilon r_0 \exp(\lambda r_0^{-1} w_* t + in\varphi) J_n(\xi_{nk} r r_0^{-1}).$$

Here ε is a small parameter, λ is a complex parameter, n is a natural number, and ξ_{nk} is the k-th root of the equation

$$\frac{d}{dx} J_n(x) = 0.$$

Carrying out the standard procedure of linearizing the equations of hydrodynamics and kinetics in the vicinity of the basic solution and separating variables, one can show that the problem of finding small perturbations reduces to the following boundary-value problem for a system of ordinary linear differential equations.

Find a set of functions $u_i(x)$, $1 \le i \le 5$, satisfying, on the interval $(0, 1)$, the system of equations:

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$$\frac{du_1}{dx} = \left[-\frac{\lambda(1-cx)g(x)}{c(\gamma+1)x^{2m}} - \frac{2}{(\gamma+1)x} + \frac{c}{1-cx}\right]u_1 +$$
$$+\left[\frac{\lambda(1+\gamma cx)g(x)}{c(\gamma+1)x^{2m}} + \frac{\gamma}{(\gamma+1)x} - \frac{(m-1)c\gamma}{1-cx} - \frac{ac\gamma}{(1+\gamma cx)(1-cx)^2}\right]u_2 -$$
$$-\frac{\xi_{nk}(1+\lambda cx)g(x)}{c(\gamma+1)x^{2m}}u_3 + \left[-\frac{1}{(\gamma+1)x} - \frac{c}{1-cx} + \frac{ac}{(1+\gamma cx)(1-cx)^2}\right]u_4 -$$
$$-\frac{mc}{x(1-cx)}u_5;$$

$$\frac{du_2}{dx} = \left[\frac{\lambda(1-cx)g(x)}{c(\gamma+1)x^{2m}} + \frac{2}{(\gamma+1)x} - \frac{c}{1+\gamma cx}\right]u_1 +$$
$$+\left[-\frac{\lambda(1-cx)g(x)}{c(\gamma+1)x^{2m}} - \frac{\gamma}{(\gamma+1)x} + \frac{(m-1)c\gamma}{1-cx} + \frac{ac\gamma}{(1+\gamma cx)^2(1-cx)}\right]u_2 +$$
$$+\frac{\xi_{nk}(1-cx)g(x)}{c(\gamma+1)x^{2m}}u_3 +$$
$$+\left[\frac{1}{(\gamma+1)x} + \frac{c}{1+\gamma cx} - \frac{ac}{(1+\gamma cx)^2(1-cx)}\right]u_4 + \frac{mc}{x(1+\gamma cx)}u_5;$$

$$\frac{du_3}{dx} = \frac{\xi_{nk}(1-cx)(1+\gamma cx)g(x)}{x^{2m-1}}u_2 + \frac{\lambda g(x)}{x^{2m-1}}u_3;$$

$$\frac{du_4}{dx} = \left[\frac{\lambda(1-cx)g(x)}{c(\gamma+1)x^{2m}} + \frac{2}{(\gamma+1)x} - \frac{c}{1-cx}\right]u_1 +$$
$$+\left[-\frac{\lambda(1+\gamma cx)g(x)}{c(\gamma+1)x^{2m}} - \frac{\gamma}{(\gamma+1)x} + \frac{(m-1)c\gamma}{1-cx} + \frac{ac\gamma}{(1+\gamma cx)(1-cx)^2}\right]u_2 -$$
$$-\frac{\xi_{nk}(1-cx)g(x)}{c(\gamma+1)x^{2m}}u_3 +$$
$$+\left[\frac{\lambda g(x)}{x^{2m-1}} + \frac{1}{(\gamma+1)x} + \frac{c}{1-cx} - \frac{ac}{(1+\gamma cx)(1-cx)^2}\right]u_4 + \frac{mc}{x(1-cx)}u_5;$$

$$\frac{du_5}{dx} = -2u_1 + \left[2(m-1)\gamma + \frac{2a\gamma}{(1-cx)(1+\gamma cx)}\right]u_2 -$$
$$-\frac{2a}{(1-cx)(1+\gamma cx)}u_4 + \left[\frac{\lambda g(x)}{x^{2m-1}} + \frac{2m-1}{x}\right]u_5, \tag{2}$$

where

$$g = \frac{1}{\sigma}(1+\gamma cx)^{-m+1}\exp\left[a(1+\gamma cx)^{-1}(1-cx)^{-1}\right],$$

and with the following boundary conditions:

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1) the condition of boundedness of the perturbations of pressure, density, and the velocity vector, and of the vanishing of the concentration perturbation at the Jouguet point, which in terms of the functions $u_i(x)$ means:

$$\text{as } x \to +0 \qquad u_k(x) \text{ are bounded}, \quad 1 \le k \le 4, \qquad x u_5(x) \to 0; \tag{3}$$

2) the conditions of conservation of mass, momentum, energy, and concentration in passing through the discontinuity, which means:

$$\text{for } x = 1 \qquad u_1 = \frac{2(2+c-\gamma c)}{(\gamma+1)(1-c^2)}\lambda g(1) - \frac{c}{(1-c)^2},$$
$$u_2 = -\frac{4}{(\gamma+1)(1+\gamma c)}\lambda g(1) + \frac{c}{(1-c)(1+\gamma c)}, \qquad u_3 = 2c\xi_{nk} g(1),$$
$$u_4 = -\frac{4(1-\gamma c)}{(\gamma+1)(1-c^2)}\lambda g(1) + \frac{c}{(1-c)^2}, \qquad u_5 = \frac{2c}{1-c}. \tag{4}$$

The value of λ for which problem (2)–(4) is solvable will be called an eigenvalue. The presence among the set of eigenvalues of at least one with $\operatorname{Re}\lambda > 0$ means instability of the basic solution (1).

The character of the boundary-value problem (2)–(4) is essentially different in the cases $\operatorname{Re}\lambda > 0$ and $\operatorname{Re}\lambda < 0$. Namely, it turns out that for $\operatorname{Re}\lambda > 0$ system (2) has four linearly independent solutions satisfying condition (3); for $\operatorname{Re}\lambda < 0$ system (2) has one solution satisfying condition (3). It is easy to see that in the latter case the eigenvalue problem is overdetermined.

Directly from system (2) and conditions (3), (4) it is seen that the set of eigenvalues is symmetric with respect to the axis $\operatorname{Im}\lambda = 0$. Investigation of the asymptotic behavior of the solutions of system (2) as $\lambda \to \infty$ shows that, for fixed n and k, all eigenvalues are concentrated in a finite region of the λ-plane.

Below are given the results of a numerical solution of problem (2)–(4) on an electronic computer of the Computing Center of the Siberian Branch of the Academy of Sciences of the USSR. The computation was carried out for fixed values of the parameters $\gamma = 1.2$, $c = 0.7925$, $m = 1$, $a = 8$, and variable $\delta = d r_0^{-1}$, where d is the effective width of the zone of chemical reaction ([1]).

The dependence of λ on ξ_{nk} and δ is as follows:

$$\lambda = \xi_{nk} f(\delta \xi_{nk}),$$

where f is a certain complex-valued (multi-valued) function. Fixing the value $k = 1$, we denote $\lambda_n = \xi_{n1} f(\delta \xi_{n1})$ and put $n = 1$.

Fig. 1: Dependence of the quantity λ_1 on δ.

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For $\delta = 0.475$ there exists an eigenvalue with $\operatorname{Re}\lambda_1 = 0$, $\operatorname{Im}\lambda_1 = 1.887$. As δ increases, the value $\operatorname{Im}\lambda_1$ decreases monotonically; the value $\operatorname{Re}\lambda_1$ first increases, and then begins to decrease and, finally, at $\delta = 1.35$ becomes equal to zero. The dependence of the quantity λ_1 on δ is shown in Fig. 1 by the solid line.

For $\delta = 0.557$, in the half-plane $\operatorname{Re}\lambda > 0$ there appears another eigenvalue, which disappears already at $\delta = 2.15$. In Fig. 1 this eigenvalue corresponds to the dashed line.

For $\delta < 0.475$, in the half-plane $\operatorname{Re}\lambda > 0$ there are no eigenvalues with $\operatorname{Im}\lambda_1 \neq 0$. But if one passes from $n = 1$ to $n = 2$, then we obtain $\lambda_2 = \xi_{21} f(\delta\xi_{11} \cdot \xi_{21}\xi_{11}^{-1})$, and for $\delta = 0.475$ we have $\operatorname{Re} f = 0.069 > 0$. With further decrease of δ, the value $\operatorname{Re}\lambda_2$ decreases monotonically and at $\delta = 0.287$ turns to zero. In Fig. 1 this dependence is shown by the dash-dotted line.

At $\delta = 0.287$ a transition occurs from $n = 2$ to $n = 3$. With further decrease of δ, the process described can be continued without bound.

The computations showed that, in addition to the eigenvalues indicated, over a very wide range of variation of δ there exists an eigenvalue with $\operatorname{Im}\lambda = 0$. Its dependence on ξ_{nk} and δ is well approximated by the formula

$$\lambda = 0.204\delta^{-1} - 0.145\xi_{nk}, \qquad 0.1 \leqslant \delta\xi_{nk} \leqslant 1.$$

A mathematical investigation of the set of eigenvalues over the entire interval $0 < \delta < \infty$ presents great difficulties; however, it is possible to prove the following facts.

If $a > 0.157\gamma^{-1}(\gamma + 1)^2$, then there exists a δ_1 such that for $\delta \leqslant \delta_1$ there is an eigenvalue with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda = 0$.

If $c < 1/2$, then for any $\varepsilon > 0$ there exists a δ_2 such that for $\delta \geqslant \delta_2$ there are no eigenvalues in the region $\operatorname{Re}\lambda \geqslant 0$, $|\lambda| \geqslant \varepsilon$.

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