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Abstract

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CUBATURE FORMULAS ON THE SPHERE

(Presented by Academician S. L. Sobolev on 12 II 1963)

In this article we report the results of recently completed work on approximate integration over the surface of a sphere. Formulas possessing the symmetry of regular polyhedra will apparently prove advantageous: their systematic study requires the use of results from the theory of group characters ([1,2]). The author of the present work has also applied these methods ([3]) and obtained several invariant integration formulas, exact for spherical polynomials of degree not exceeding n. The values of n are not large, but the efficiency is high, i.e. $E = (n+1)^2/3N$, where N is the number of nodal points and $(n+1)^2$ is the number of linearly independent spherical polynomials of degree not exceeding n, is close to unity. Some of these formulas were obtained by S. L. Sobolev ([2]); D. G. Kendall showed how to derive others from cubature formulas for the sphere of the type given by B. A. Ditkin ([4]) and, later, by P. C. Hammer and A. H. Stroud ([5]). The following results are apparently new:

Group	G_{VIII}	G_{VIII}	G^*_{VIII}	G_{XX}
n	4	8	11	17
N	24	30	50	72
E	0.89	0.90	0.96	1.04

The weights and locations of the nodal points have been determined by us. Equally efficient formulas apparently also exist for larger values of n, but the computations become difficult. If the nodal points are not chosen carefully, then E tends to $1/3$ as n tends to infinity.

D. G. Kendall obtained another set of formulas, from which one can choose a formula exact up to an arbitrarily large (odd) value of n. For this formula $\frac{1}{2}(n+1)^2$ points are chosen so that $E = 2/3$. These Kendall formulas are Cartesian-product formulas, possessing certain practical advantages. They are obtained from results of Pearce ([5]) concerning the surface of a sphere. The same method had earlier been used by Ditkin ([4]) for the ball.

2. Another approach to the problem of integration on the sphere is as follows. Suppose a class of integrand functions $f(\theta, \Phi)$ is considered, for which the formula is applicable as a stochastic process defined on the surface of the sphere. If this process is invariant with respect to all rotations, then it is described by a covariance function

$$\mathcal{E}\left[f(\theta_r, \Phi_r) f(\theta_s, \Phi_s)\right] \equiv \Gamma(\cos\gamma_{rs}) = \frac{1}{4\pi} \sum_{m=0}^{\infty} (2m+1) \lambda_m P_m(\cos\gamma_{rs}),$$

where

$$\cos\gamma_{rs} = \cos(\theta_r - \theta_s) - 2\sin\theta_r \sin\theta_s \sin^2\frac{1}{2}(\Phi_r - \Phi_s),$$

and all λ_m are nonnegative ([7]). Without loss of generality, we may assume that $\mathcal{E}\left[f(0, \Phi)\right] = 0$, provided that the formula integrates constants correctly.

In this case the quadratic measure of inaccuracy in the formula

$$\sum_{r=1}^{k} \alpha_r f(\theta_r, \Phi_r) \sim \int f(\theta, \Phi)\, dS,$$

where $\sum \alpha_r = \int dS = 4\pi$, is

$$I^* \equiv \mathcal{E}\left[\sum_{r=1}^{k} \alpha_r f(\theta_r, \Phi_r) - \int f(\theta, \Phi)\, dS\right]^2 =$$

$$= \sum_{r=1}^{k} \sum_{s=1}^{k} \alpha_r \alpha_s \Gamma(\cos\gamma_{rs}) - 4\pi\lambda_0.$$

In an addendum to our paper[3] reasons will be given for considering the case

$$\lambda_m = \rho^m \quad (0 < \rho < 1).$$

Thus,

$$\Gamma(\cos\theta) = \frac{1}{4\pi} \sum_{m=0}^{\infty} (2m+1)\rho^m P_m(\cos\theta) = \frac{1}{4\pi} \frac{1-\rho^2}{[1 - 2\rho\cos\theta + \rho^2]^{3/2}}.$$

If ρ is small, the higher harmonics decay rapidly, and as $\rho \downarrow 0$ $f(\theta, \Phi)$ turns into a random constant on the sphere. Precisely the opposite case, when ρ is close to 1, is better suited to the practical situation in which the expansion of the integrand functions in spherical harmonics does not terminate after a finite number of terms. As $\rho \uparrow 1$, $\Gamma(\cos\theta)$ turns into the δ-function, and $f(\theta, \varphi)$ is an uncorrelated invariant field. If, however, ρ takes a value strictly less than unity,

$$\Gamma(\cos\theta) = \frac{1}{16\pi}(1-\rho)(2-\rho)\operatorname{cosec}^3\frac{1}{2}\theta + O[1-\rho]^3$$

provided that θ is bounded away from below. Thus, for ρ close to 1, the optimal choice of nodal points for an equally weighted integration formula does not depend sharply at all on the exact value of ρ. The optimal choice of the arrangement of nodal points is the one that gives a minimum of $\sum_{r<s} \operatorname{cosec}^3 \frac{1}{2}\gamma_{rs}$. This variational problem has a dynamical interpretation, namely: it can be reduced to solving the problem of the position of stable equilibrium of particles placed on the surface of a sphere and subject to a known law of mutual repulsion.

3. Both methods described in this paper can evidently be generalized to the case of spheres in a space of n dimensions. The author hopes to show that some well-known problems of mathematical statistics are precisely problems of spherical integration in a modified form.

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Note: Figure translations are in progress. See original paper for figures.