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Abstract

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MATHEMATICS

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COMPLEX VARIABLES IN PROBLEMS WITH NUISANCE PARAMETERS AND FINITE-RANK SUFFICIENT STATISTICS

In a note (^{1}) I indicated the application of analytic continuation with respect to a parameter for the investigation of the Behrens–Fisher problem in its simplest form.* It is also stated there that the introduction of complex variables has a rather broad range of applications in statistical problems of hypothesis testing and interval estimation in the presence of nuisance parameters. Here we shall outline several problems that can be successfully studied by the indicated method.

The simplest objects for the application of such a method will be systems consisting of a finite number r of independent repeated samples $O_1, O_2, \dots, O_r$, of respective sizes $n_1, n_2, \dots, n_r$. The sample O_i consists of n_i observations $x_{i1}, x_{i2}, \dots, x_{in_i}$ of a one-dimensional random variable having probability density, with respect to the measure $\mu_i(x)$, equal to $p_i(x, \theta_{i1}, \dots, \theta_{ik_i})$. Here $\theta_{i1}, \dots, \theta_{i,k_i}$ are k_i scalar parameters varying independently in a k_i-dimensional parallelepiped of Euclidean space E_{k_i}. Thus, the distributions of x_{il} are dominated by the measure $\mu_i(x)$ for all admissible values of the parameters θ_{ik}.

Next, suppose that the family of densities $p_i(x, \theta_{i1}, \dots, \theta_{ik_i})$ admits a system of scalar sufficient statistics of bounded rank for the parameters $\theta_{i1}, \dots, \theta_{ik_i}$, and that this rank does not depend on the sample size n_i. In this case, as is known, under fairly general conditions ($^{2-4}$) the densities $p_i(x, \theta_{i1}, \dots, \theta_{ik_i})$ form an exponential family.** The totality of observations of all our samples will have density

$$p(X, \theta) = \exp\left(C_0(\theta) + \sum_{i=1}^{r} \sum_{j=1}^{k_i} V_{ij}\theta_{ij}\right), \tag{1}$$

where $X = (x_{11}, \dots, x_{rn_r})$; $\theta = (\theta_{11}, \dots, \theta_{rk_r})$; V_{ij} are sufficient statistics, with V_{ij} depending only on the elements of O_i; the density is taken with respect to the product measure $\mu(X)$ for $\mu_i(x)$; $C_0(\theta)$ is a function only of the parameters.

The problems under study are tests of statistical hypotheses H_0:

$$Q_1(\theta) = 0, \qquad Q_2(\theta) = 0, \dots, Q_\nu(\theta) = 0, \tag{2}$$

where $\nu < \sum_{i=1}^{r} k_i$, and $Q_1(\theta), \dots, Q_\nu(\theta)$ are given homogeneous polynomials in the parameters of arbitrary degrees. It is proposed to determine whether there exists a nontrivial measurable function $g(T_1, \dots, T_s)$, where $T_1, \dots, T_s$ are given statistics of the totality of samples $O_1, \dots, O_r$, such that its distribution does not depend on the parameters θ under the hypothesis H_0, i.e., under the relations (2). Such a function g will be called a test (a trivial and completely useless test is, obviously, any constant).

* The conditions on the test lines introduced in note (^{1}) can be considerably weakened.

** Which we shall regard as naturally parametrized.

The question is reduced to the following:

Let $\psi(g)$ be a function of one variable, bounded on the whole axis and piecewise continuous. Can it be the case, for any such $\psi(g)$, that the integral relation

$$\int_{\mathfrak{X}} \dots \int \psi(g(T_1, \dots, T_s)) \exp\left(\sum_{i=1}^{r} \sum_{j=1}^{k_i} V_{ij}\theta_{ij} \right) dX = C_\psi \exp[-C_0(\theta)] \tag{3}$$

holds for all values of θ lying on the manifold (2)? Here C_ψ is a constant depending only on ψ; $\mathfrak{X}$ is the sample space (Euclidean space of $K = \sum_{i=1}^{r} k_i$ measurements); the product measure $d\mu(X)$ has, for simplicity, been replaced by dX.

Let $\theta = (\theta_{11}, \dots, \theta_{rk_r})$ be some point of the manifold (2). For any $\omega > 0$, the point $\theta/\omega = (\vartheta_{11}, \dots, \vartheta_{rk_r})$ will also be a point of this manifold.

Consider analytic continuations, into the complex planes, of relation (3) with respect to the parameters θ, subject to (2); suppose that they can be carried out along some ray $(\vartheta_{11}\omega, \dots, \vartheta_{rk_r}\omega)$, where the ϑ_{ij} are arbitrary complex numbers. Let $a > 0$, $b > 0$ be positive constants. Multiply expression (3) by $\omega^b e^{-a\omega}$ and integrate with respect to ω from 0 to ∞; suppose that the resulting integrals converge absolutely and uniformly in some region of admissible values of ϑ. Relation (3) then becomes the equality

$$\int_{\mathfrak{X}} \dots \int \frac{\psi(g(T_1, \dots, T_s))\, dX}{\left(a - \sum_{i=1}^{r} \sum_{j=1}^{k_i} V_{ij}\vartheta_{ij}\right)^{b+1}} = C_\psi f(\vartheta, a, b), \tag{4}$$

where $f(\vartheta, a, b)$ does not depend on ψ, and ϑ is subject to (2).

We now carry out the analytic continuation of the new relation (4) into the complex planes with respect to the parameters ϑ under condition (2). We choose the function $\psi(g)$ in the form of an impulse function: $\psi(g) = 1$ for $C \leq g \leq C + \Delta C$; $\psi(g) = 0$ otherwise. The regions $\psi(g) = 1$ of the space $\mathfrak{X}$ will be called test layers. Let $\vartheta = (\vartheta_{11}, \dots, \vartheta_{rk_r})$ be a real vector of the manifold (2) such that

$$a - \sum_{i=1}^{r} \sum_{j=1}^{k_i} V_{ij} \vartheta_{ij} = 0. \tag{5}$$

The corresponding surface (5) in the space $\mathfrak{X}$ will be called the critical surface, or the **critical set**, of our family of measures on $\mathfrak{X}$. Studying the possible positions of test layers relative to the critical set, under certain smoothness conditions on the test layers, in many cases makes it possible to detect the nonexistence of tests of the given form $g(T_1, \dots, T_s)$, while in other cases it provides a means of constructing approximate tests. One of the simplest and roughest approaches to such a study is the following: we bind the parameters ϑ in (4) on the manifold (2) so as to obtain a function of one complex variable ϑ'. We regard the test layers as bounded by a finite number of piecewise analytic surfaces. Let ϑ' be a singular point for $f(\vartheta, a, b)$, say a pole or a branch point. Substituting ϑ' into (5), we obtain a critical set. If it is imaginary, or if it is real but the test layer $C \leq \psi(g) \leq C + \Delta C$ exists for some C and ΔC and is separated from the critical set by a distance $\delta > 0$, then it is easy to prove that the test does not exist. If the critical set is piecewise-analytic, then, by investigating its intersections with test layers, it is not difficult to establish the nonexistence of tests in many cases. If the point ϑ' is a regular point of $f(\vartheta, a, b)$, then it turns out that, for the existence of a piecewise-analytic test, the layer must not touch the corresponding critical region (but may contain it completely). To detect the nonexistence of the given test in this case, it is sufficient, generally speaking, to find a test layer touching the critical region (5) corresponding to the given point ϑ' (a regular point of $f(\vartheta, a, b)$).

In particular cases, one or another variant of the scheme described is convenient. For the homogeneous Behrens—Fisher problem [1], we obtain the relation (in the notation of [1]):

$$\iint_{\Omega} \psi(g(\xi, \eta)) \frac{\eta^{n_1 - 2}\, d\xi\, d\eta}{\left(\vartheta^2 + \vartheta(1 + \xi^2 + \eta^2) + \eta^2\right)^N} = C_\psi \vartheta^{-n_2/2} (1 + \vartheta)^{-N + 1/2}. \tag{6}$$

Here there is analytic continuation in ϑ to the whole complex plane with a cut $(-\infty, 0)$. The critical regions are obtained for $\vartheta \leqslant 0$. They are formed by systems of ellipses

$$\frac{\xi^2}{D - 1} + \frac{\eta^2}{D} = 1 \quad (D > 1),$$

hyperbolas

$$\eta^2\left(\frac{1}{D}-1\right)-\xi^2=1-D \quad (D<1),$$

and the straight lines $\xi=0,\ \eta=0$.

A piecewise-analytic test $g(\xi,\eta)$, taking all values from the segment $[0,M]$ for $M>0$, will touch by a layer the ellipse-critical region

$$\frac{\xi^2}{D-1}+\frac{\eta^2}{D}=1$$

for a suitable $D>1$, or will intersect the axis of abscissas with it at one point and therefore, as it turns out, cannot exist.

Let us also consider, as an example, the problem of k normal samples (see, for example, [5], p. 296, with the same notation). Here the density with respect to Lebesgue measure is

$$p=\frac{1}{(2\pi)^{N/2}}\frac{1}{\prod_{i=1}^{k}\sigma_i^{n_i}}\exp\left(-\sum_{i=1}^{k}n_i\frac{(\bar{x}_i-\mu_i)^2+s_i^2}{2\sigma_i^2}\right). \tag{7}$$

We may, for example, consider the hypothesis

$$H_0:\ \mu_1=\mu_2=\cdots=\mu_k=\mu$$

and seek a test $g(\bar{x}_i,s_i)$, depending only on sufficient statistics, piecewise-analytic (with possible exceptions of measure 0), and similar with respect to σ_i^2.

To investigate the question of the existence of such a test, it is convenient to set

$$\frac{1}{2\sigma_i^2}=\vartheta_i\omega$$

and to carry out the integration of expression (3) indicated above.

We arrive at the expression:

$$\int_{\mathfrak{X}}\cdots\int\frac{\psi(g(\bar{x}_i,s_i))\,dX}{\left[a+\left(\sum_{i=1}^{k}n_i\vartheta_i((\bar{x}_i-\mu_i)^2+s_i^2)\right)\right]^{b+1}}=C_\psi C(a,b)\prod_{i=1}^{k}\vartheta_i^{-1/2} \tag{8}$$

for sufficiently large b. The critical regions here will be surfaces of the second order, and it is possible to carry out a successful investigation of the possibility of a test. The method set forth can also be applied to the construction of approximate tests and to the investigation of confidence intervals.

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Note: Figure translations are in progress. See original paper for figures.

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