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V. N. KALININ

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Abstract

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V. N. KALININ

ON THE EXISTENCE OF AN ABSOLUTE MINIMUM IN ONE PROBLEM OF THE THEORY OF OPTIMAL PROCESSES

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In the present note an attempt is made to justify direct methods as applied to one problem in the theory of optimal control. Theorem 2 is given, adjacent to the known results of Tonelli in the classical calculus of variations ([1]), and establishing conditions for the existence of an absolute minimum in the problem posed and of a minimizing sequence converging to an element realizing this absolute minimum.

1. Definition 1. We shall call the class of admissible controls the set of all Lebesgue-measurable vector functions $u(t)$, defined on the interval $a = [0, \tau]$ and, for almost all $t \in a$, mapping this interval into a closed domain G of the r-dimensional vector space R. We denote the class of admissible controls by U.

Definition 2. Suppose an operation $A(u, x)$ is given, defined on the direct product of U and the space C_a of vector continuous functions with values in the n-dimensional vector space H, defined on the interval a, and having values in C_a. If the indicated operation is bounded and, uniformly with respect to $u \in U$, is a contraction operation in C_a, then we shall say that it defines a controlled process. By a controlled process is meant the fixed point of the indicated operation for a given $u \in U$, which is the solution of the functional equation

$$x = A(u, x). \tag{1}$$

Let us combine all operations of the indicated kind into one class, denoted below by Δ. To each operation of this class there corresponds a single-valued and bounded mapping of the class of admissible controls U onto some set P of controlled processes in C_a.

Definition 3. We shall call the operation that realizes the mapping $U \to P$, corresponding to the operation $A \in \Delta$, the resolvent of this operation, and denote it by $S(u)$.

Thus, the class Δ is isomorphic to some set of single-valued and bounded operations mapping the class of admissible controls U into the space C_a.

Below the following problem is considered. Suppose a functional is given

$$I(u) = \int_0^\tau f(x,u,t)\,dt, \tag{2}$$

where the function $f(x,u,t)$ is understood to be such that the indicated integral exists (in the Lebesgue sense), is bounded and single-valued for any admissible control $u \in U$ and the corresponding controlled process $x \in P$ (with $A \in \Delta$). It is required to establish conditions under which, in the class of admissible controls U, there exists a control u^* delivering to the functional (2) an absolute minimum in this class. We shall call this control optimal.

2. Definition 4. Let the integrand $f(x,u,t)$ be defined for $x \in H$, $u \in G$, $t \in a$ and in the indicated domain: a) together with the partial derivatives $f_{u_\rho}(x,u,t)$ and $f_{u_\rho u_\sigma}(x,u,t)$, $\rho, \sigma = 1, \dots, r$, for almost all $t \in a$ is continuous as a function of u and x and for all u and x (with $|x| < \infty$)

measurable and almost everywhere on α bounded with respect to t; b) the real quadratic form

$$\sum_{\rho,\sigma=1}^{r} f_{u_\rho u_\sigma}(x,u,t)\,\beta_\rho \beta_\sigma$$

is nonnegative for all $x \in H$, $u \in G$ and for almost all $t \in \alpha$. Then we shall say that the functional (2) is quasiregular in the class U. If the metric in U is induced by its embedding in L^p $(1 < p < \infty)$, then the following theorem can be proved:

Theorem 1. *If the functional $I(u)$ is quasiregular in the class of admissible controls U and the resolvent $S(u)$ is strongly continuous for $u \in U$, then $I(u)$ is weakly lower semicontinuous on U.*

The proof of this theorem is based on considering the identity:

$$\begin{aligned} I(u_0) = I(u_\nu) &+ \int_0^\tau [f(x_0,u_0,t) - f(x_\nu,u_0,t)]\,dt \\ &+ \int_0^\tau [f(x_\nu,u_0,t) - f(x_\nu,u_\nu,t)]\,dt, \end{aligned} \tag{3}$$

where the sequence $\{u_\nu\} \in U$ converges weakly to $u_0 \in U$, $x_\nu = S(u_\nu)$, $x_0 = S(u_0)$, and on estimating the integrals occurring on the right-hand side. It can be shown that, under the assumptions of the theorem,

$$I(u_0) \le \lim_{\nu\to\infty} I(u),$$

which also means the weak lower semicontinuity of $I(u)$ on U.

3. Denote by Δ_0 the set of operations from Δ having a strongly continuous resolvent. Since $U \in L^p$ for $1 < p < \infty$ is evidently bounded and weakly closed and, by virtue of the regularity of L^p $(1 < p < \infty)$, weakly compact (the Gantmacher–Shmulyan–Eberlein theorem ([2])), it follows, taking into account Theorem 1 and known theorems of nonlinear functional analysis ([2]), that

Theorem 2. *If $A \in \Delta_0$ and the functional $I(u)$ (2) is quasiregular in U, then in the class of admissible controls there exists a control u^* giving it an absolute minimum in this class (an optimal control).*

Hence follows the existence in the class U of a minimizing sequence converging to u^* at least weakly. As for the set Δ_0, the following theorem is useful for singling it out from the class Δ.

Theorem 3. *In order that $A \in \Delta_0$, it is sufficient that $A \in \Delta$ and that, for every $x \in C_\alpha$, A be strongly continuous with respect to u on U.*

We note that from the condition of the theorem there follows, generally speaking, not only the strong continuity of the resolvent $S(u)$, but also its complete continuity. Indeed, by the theorem of Tsitlanadze ([3]), a strongly continuous operation defined in a space with a weakly compact ball is completely continuous; thus the following is established:

Corollary. *If the operation $A \in \Delta_0$, then its resolvent $S(u)$ is completely continuous.*

Moreover, evidently, the class P of controlled processes in C_α is compact, i.e. it is a set of uniformly bounded and equicontinuous functions.

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2. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.

3. E. S. Tsitlanadze, *Mat. sbornik*, **29** (71), no. 1, 3 (1951).

Note: Figure translations are in progress. See original paper for figures.