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Abstract

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MATHEMATICS

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ON THE ANALYTIC THEORY OF TESTS FOR THE BEHRENS–FISHER PROBLEM

Let $x_1, \dots, x_{n_1} \in N(a_1, \sigma_1)$ and $y_1, \dots, y_{n_2} \in N(a_2, \sigma_2)$ be two independent repeated normal samples with four unknown parameters. We shall consider nonrandomized tests for the Behrens–Fisher problem (testing the hypothesis $a_1 = a_2$), similar with respect to σ_1 and σ_2 and satisfying the first three of the four known axioms of A. Wald (^{1}). Namely, we consider functions of the observations $G(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2})$, measurable with respect to the probability measures induced by the samples for all values of the parameters, and critical regions $\mathfrak{R}_C : \; G(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \geq C$, which for all C are subject to the following three axioms.

1. The regions $\mathfrak{R}_C$ lie in the space of sufficient statistics $\bar{x}, \bar{y}, s_1^2 =$

$$= \frac{1}{n_1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2, \quad s_2^2 = \frac{1}{n_2} \sum_{j=1}^{n_2} (y_j - \bar{y})^2,$$

i.e., if the sample $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathfrak{R}_C$, then $\mathfrak{R}_C$ also contains any other sample $(x'_1, \dots, x'_{n_1}, y'_1, \dots, y'_{n_2})$ for which $\bar{x}' = \bar{x}, \; \bar{y}' = \bar{y}, \; s_1'^2 = s_1^2, \; s_2'^2 = s_2^2$.

2. If the sample $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathfrak{R}_C$, then for any c the sample $(x_1 + c, \dots, x_{n_1} + c, y_1 + c, \dots, y_{n_2} + c) \in \mathfrak{R}_C$.
3. If the sample $(x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}) \in \mathfrak{R}_C$, then for any $k \neq 0$ the sample $(kx_1, \dots, kx_{n_1}, ky_1, \dots, ky_{n_2}) \in \mathfrak{R}_C$.

From these three axioms it follows that the function G has the form $G = g(\xi, \eta)$, where $\xi = \dfrac{\bar{x} - \bar{y}}{s_2}$, $\eta = \dfrac{s_1}{s_2}$. In what follows we shall call the function $g(\xi, \eta)$ itself a test; it is a measurable function. The statistics ξ and η range over the half-plane $\Omega = (-\infty < \xi < \infty, \; 0 \leq \eta < \infty)$, where the samples induce the family of densities

$$C_{n_1, n_2} \vartheta^{n_2/2} (1 + \vartheta)^{-N-1/2} \frac{\eta^{n_1 - 1}}{[\vartheta^2 + \vartheta(1 + \xi^2 + \eta^2) + \eta^2]^N},$$

where

$$N = \frac{n_1 + n_2 - 1}{2}, \qquad \vartheta = \frac{n_2}{n_1}\frac{\sigma_1^2}{\sigma_2^2}.$$

Definition 1. We shall call a test $g(\xi, \eta)$ **regularly varying** if, whatever the semicircle $K \subset \Omega$ with center at the origin,

$$\operatorname{vrai}\max_K g(\xi, \eta) < \operatorname{vrai}\max_\Omega g(\xi, \eta), \tag{1}$$

or

$$\operatorname{vrai}\min_K g(\xi, \eta) > \operatorname{vrai}\min_\Omega g(\xi, \eta).$$

Otherwise the test $g(\xi, \eta)$ will be called **singularly varying**.

This note is devoted to the question of the existence of tests of either type under certain additional restrictions. In doing so we use the method of analytic continuation with respect to the parameter ϑ, introduced in (^{2}).

Theorem 1. *A regularly varying similar test for two samples of different parity cannot exist.*

More cumbersome is the investigation of singularly varying similar tests. We note that a similar test of this type with critical region volume ···

zones equal to 1/2 exists. This test is

$$g(\xi, \eta) = \begin{cases} 1, & \xi \geqslant 0, \\ 0, & \xi < 0; \end{cases}$$

the boundary of the critical region here passes through the point $(0, 0)$.

Let $A \subset \Omega$ be a measurable set. Consider the family of semicircles $K \subset \Omega$ with center at the point O, the origin of coordinates. Let us call $\operatorname{vrai}\operatorname{dist}(O, A)$ the exact upper bound of the radii of the semicircles K that intersect A in a set of measure zero. Let Π_n be the strips $0 \leqslant \eta \leqslant 1/n$; put $A_n = \mathfrak{K}_C \cap \Pi_n$ and

$$\rho_0 = \lim_{n \to \infty} \operatorname{vrai}\operatorname{dist}(O, A_n).$$

Lemma. For a similar test $g(\xi, \eta)$, $\rho_0 < \infty$.

For any test indicated above, $\rho_0 = 0$.

Theorem 2. There is no similar test $g(\xi, \eta)$ under the conditions:

I. $\rho_0 > 0$.

II. For some $\varepsilon > 0$ in the strip between the confocal semiellipses

$$\frac{\xi^2}{\rho_0^2 + \varepsilon} + \frac{\eta^2}{\rho_0^2 + \varepsilon + 1} = 1, \quad \eta \geqslant 0,$$

the boundary of the critical region $\mathfrak{K}_C :\ g(\xi, \eta) = C$ (for some C) consists of a finite number of piecewise quasianalytic curves.

III. The sample sizes are not equal.

We outline the proof of Theorem 1. Suppose that a regularly varying similar test exists. Consider the integral relation expressing the similarity of the test (see $(^2)$, formula (1)):

$$\iint_\Omega \Psi[g(\xi, \eta)] \frac{\eta^{n_1 - 2}\, d\xi\, d\eta}{\left[\vartheta^2 + \vartheta(1 + \xi^2 + \eta^2) + \eta^2\right]^N} = C_\Psi \vartheta^{-n_2/2} (1 + \vartheta)^{-N + 1/2}, \quad (2)$$

where Ψ is an arbitrary bounded continuous function and C_Ψ is a constant depending on Ψ. Note that for samples of different sizes N is an integer. Let $\vartheta_1 = \vartheta_1(\xi, \eta)$ and $\vartheta_2 = \vartheta_2(\xi, \eta)$ be the roots of the equation

$$\vartheta^2 + \vartheta(1 + \xi^2 + \eta^2) + \eta^2 = 0.$$

It turns out that $-1 \leqslant \vartheta_1 \leqslant 0$ and $\vartheta_2 \leqslant -1$. Take a semicircle $K \subset \Omega$ with radius greater than 1 and put

$$\Psi(z) = \begin{cases} 1, & z > \operatorname*{vrai\,max}_K g(\xi, \eta), \\ 0, & z < \operatorname*{vrai\,max}_K g(\xi, \eta), \end{cases}$$

assuming, for definiteness, that (1) holds.

By the regular variation of $g(\xi, \eta)$, the constant C_Ψ in formula (2) will be different from zero. Moreover, if $(\xi, \eta) \in K$, then there exists $\Delta > 0$ such that $\vartheta_2 < -1 - \Delta$. Therefore an analytic continuation of relation (2) with respect to the parameter ϑ is possible onto the plane with cuts $\operatorname{Im} \vartheta = 0$, $-1 \leqslant \operatorname{Re} \vartheta \leqslant 0$, $\operatorname{Re} \vartheta \leqslant -1 - \Delta$, and

$$\vartheta = -1 - \frac{\Delta}{2} + \frac{\Delta}{2} e^{i\varphi}, \quad -\pi \leqslant \varphi \leqslant 0.$$

Now let $\vartheta = -1 - \frac{\Delta}{2}$. Compare the right- and left-hand sides of (2). We see that, whereas the right-hand side is an imaginary number, the left-hand side is a real number. This contradiction proves Theorem 1.

The proof of Theorem 2 is considerably more difficult. Here the cut in the complex ϑ-plane is made along the negative real axis. The value chosen is $-\rho_0^2 - 1 + i\zeta$, where $\zeta > 0$ is small. As $\zeta \to 0$, the right-hand side of (2) has a finite limit. The modulus of the left-hand side, however, under the conditions of Theorem 2, turns out to tend to ∞. The conditions of Theorem 2 can be considerably weakened.

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1. A. Wald, *Selected Papers in Statistics and Probability*, N. Y., 1955, pp. 669–695.

2. Yu. V. Linnik, DAN, 149, No. 2 (1963).

Note: Figure translations are in progress. See original paper for figures.

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