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Abstract

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INVARIANTS AND CANONICAL REPRESENTATIONS OF QUADRICS IN QUASI-ELLIPTIC SPACES

(Presented by Academician P. S. Novikov on 24 X 1962)

In the present note we find invariants and canonical representations of quadrics in quasi-elliptic spaces $(^1, ^2)$, and a substantial part of the results is proved for spaces of a more general kind.

1. Consider the Cayley-Klein space whose absolute in the projective space P_n consists of a cone of index l_0

$$\sum_{a,b} h_{ab} x^a x^b = 0, \quad a, b = 0, 1, \dots, m, \quad (1)$$

in an $(n - m - 1)$ -dimensional flat vertex in which a nondegenerate quadric of index l_1 is specified,

$$x^a = 0, \quad \sum_{u,v} g_{uv} x^u x^v = 0, \quad u, v = m + 1, \dots, n. \quad (2)$$

We shall call such a space **quasi-Euclidean** (cf. $(^1, ^2)$) and denote it by ${}_{l_0 l_1} S_n^m$. For $l_0 = l_1 = 0$ we arrive at a **quasi-elliptic** space ${}_{00} S_n^m$ (or S_n^m); if $l_0^2 + l_1^2 \neq 0$, then the space is called **quasi-hyperbolic**. For $m = 0$ we obtain a **generalized Euclidean** (or pseudo-Euclidean) space ${}_{0l_1} S_n^0$ (or ${}_l S_n^0$). For $m = l_1 = 0$ we obtain the **proper Euclidean** space S_n^0 .

In the quasi-Euclidean space ${}_{l_0 l_1} S_n^m$ consider the quadric

$$x^T M x = 0 \quad (3)$$

with coefficient matrix

$$M = \left\| \begin{array}{cc} C & B \\ B^T & A \end{array} \right\| \left\{ \begin{array}{l} m + 1 \\ n - m; \end{array} \right.$$

where T denotes transposition and x is the column matrix of the coordinates of the current point. Introduce the matrix

$$M(\mu, \lambda) = \left\| \begin{array}{cc} C - \mu H & B \\ B^T & A - \lambda G \end{array} \right\|,$$

where $H = \|h_{ab}\|$, $G = \|g_{uv}\|$, and, of course, $\det H \neq 0$ and $\det G \neq 0$. This matrix may be called the **characteristic matrix** of the quadric.

The following theorems on invariants of quadrics with respect to the motions (see (3)) of the space ${}_{l_0 l_1} S_n^m$ have been proved.

Theorem 1. *If the quadric (3) is specified up to motions of the space ${}_{l_0 l_1} S_n^m$, then the matrices $A - \lambda G$ and $M(\mu, 0)$ are determined up to congruent transformations.*

This theorem strengthens the results of (1, 4). From it there follows not only the invariance of the elementary divisors of the matrices $A - \lambda G$ and $M(\mu, 0)$, but also the invariance of the discrete characteristics of these real λ -matrices (see (5, 6)).

The invariant polynomial $\det(A - \lambda G)$ gives $n - m$ invariants of the equation of the quadric. The polynomial $\det M(\mu, 0)$ can give $m + 2$ invariants, of which $m + 1$ invariants, independent of the preceding ones, will be the ones lacking for determining the equation of the quadric. However, if the degree of the polynomial $\det M(\mu, 0)$ is less than m , then it gives $< m + 1$ independent invariants; if the rank of the matrix $\|B^T A\|$ is less than $n - m$, then this polynomial vanishes identically and provides no invariants.

Denote by q the defect of the matrix $\|B^T A\|$ with respect to rows (the number of linear dependencies among the rows of the matrix), and by p the defect of the matrix A ; $p \geq q$.

Theorem 2. The polynomial

$$m_q(\mu) = \left. \frac{\det M(\mu, \lambda)}{\lambda^q} \right|_{\lambda=0}$$

is invariant with respect to motions of the space ${}_{l_0 l_1} S_n^m$.

It can be shown that the degree of the polynomial $m_q(\mu)$ for $p \neq 0$ (for $p = q = 0$ the degree of the polynomial $m_0(\mu)$ is equal to $m + 1$) is equal to $(m + 1) - (p - q)$. In this case all coefficients of the polynomial $m_q(\mu)$ are independent of the invariants found earlier, and we obtain $(m + 2) - (p - q)$ new invariants.

Theorem 3. The coefficients of the polynomial $\det M(\mu, \lambda)$ at the terms with

$$\mu^{m+1-s} \lambda^{p-s}, \quad s = 1, 2, \dots, p - q,$$

are invariant with respect to motions of the space ${}^u_1S_n^m$.

In the case $p \neq 0$, this theorem gives $p - q - 1$ invariants independent of the preceding invariants. Theorems 1-3 in all cases provide $n + 1$ independent invariants. Discrete invariants are not involved in the count.

The proof of Theorems 2 and 3 rests essentially on the following lemma.

Lemma 1. Suppose there are two square matrices consisting of identical blocks:

$$\left\| \begin{array}{cc} X & Y \\ Z & T \end{array} \right\|, \quad \left\| \begin{array}{cc} A & B \\ C & D \end{array} \right\|,$$

where X, T, A, D are square blocks and D is nonsingular. Suppose also that the defect of the matrix $\|Z \ T\|$ with respect to rows and the defect of the matrix $\left\| \begin{array}{c} Y \\ T \end{array} \right\|$ with respect to columns are equal to q . Then, in the polynomials

$$P_1(\lambda) = \left| \begin{array}{cc} X - \lambda A & Y - \lambda B \\ Z - \lambda C & D - \lambda T \end{array} \right|, \quad P_2(\lambda) = \left| \begin{array}{cc} X & Y \\ Z & D - \lambda T \end{array} \right|,$$

the coefficients of λ^q are equal, and the terms with $\lambda^0, \lambda^1, \dots, \lambda^{q-1}$ are absent.

- Let us now consider quadrics in the quasi-elliptic space S_n^m . All conclusions are formulated under the assumption that in the space

such a coordinate system is introduced in which the equations of the absolute (1) and (2) have, respectively, the form

$$\sum_a (x^a)^2 = 0,$$

$$x^a = 0, \quad \sum_u (x^u)^2 = 0.$$

By a successive simplification of the equation of the quadric, the following theorem can be proved:

Theorem 4. *By motions of the quasielliptic space S_n^m , the equation of a quadric can be brought to such a form that the matrix of the coefficients of the quadric will have the form*

$$M_0 = \left\| \begin{array}{ccc|cc} \mu_0 & & 0 & 0 & 0 & 0 \\ & \ddots & & & & \\ 0 & & \mu_{m-(p-q)} & & & \\ \hline 0 & & & 0 & 0 & h_{p-q} \\ & & & h_1 & \ddots & \\ \hline & & 0 & 0 & h_1 & 0 \\ & & & \ddots & 0 & \\ \hline & & h_{p-q} & 0 & & 0 \\ \hline 0 & & & 0 & \lambda_{m+p+1} & 0 \\ & & & & 0 & \ddots \\ & & & & & \lambda_n \end{array} \right\|.$$

Here $\lambda_{m+p+1}, \dots, \lambda_n$ are the nonzero roots of the polynomial $\det|A - \lambda G|$; $\mu_0, \mu_1, \dots, \mu_{m-(p-q)}$ are the roots of the polynomial $m_q(\mu)$; h_1, \dots, h_{p-q} are determined on the basis of Theorem 3.

It follows from this theorem that, in the case of a quasielliptic space, the system of invariants of quadrics determined by Theorems 1–3 is complete.

For $m = 0$, the quasielliptic space becomes the proper Euclidean space S_n^0 , and Theorem 4 reduces to known results (see, for example, (7)); Theorem 3 in this case is not needed (it gives no new invariants).

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CITED LITERATURE

1. B. A. Rozenfel' d, *Tr. Mosk. matem. obshch.*, **8**, 49 (1959).
2. B. A. Rozenfel' d, T. M. Klimanova, N. D. Petsko, *Izv. vyssh. uchebn. zaved., Matem.*, No. 2 (27), 130 (1962).
3. E. U. Yasinskaya, *DAN*, **137**, No. 6, 1327 (1961).
4. L. P. Ptitsyna, L. V. Guchkova, L. V. Rumyantseva, *Dokl. AN AzerbSSR*, **16**, No. 1, 639 (1960).
5. I. M. Yaglom, *Tr. seminara po vektorn. i tenzorn. analizu*, **8**, 364 (1950).
6. Yu. B. Ermolaev, *DAN*, **132**, No. 2, 257 (1960).
7. G. E. Shilov, *Introduction to the Theory of Linear Spaces*, Moscow-Leningrad, 1952.

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