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Abstract

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A Uniqueness Theorem for Dirichlet Series in the Case of More General Behavior of the Exponents

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In note (3) a uniqueness theorem was proved for Dirichlet series with exponents $0 < \lambda_n \uparrow \infty$ for $\lambda_n \sim cn^{1/q}$, $c \neq 0, \infty$, $1 < q < 2$, and the limiting growth $M_F(x)$ was determined for the sum of a Dirichlet series, depending on the behavior of the exponents; that is, it was shown that only under the condition

$$\lim_{x\to\infty} \frac{\ln M_F(x)}{x^p} \geqslant \alpha,$$

where p and q are related by $1/p + 1/q = 1$, and α is a certain constant found, is it possible for there to exist a nonzero Dirichlet series bounded on the real axis.

In the present work an analogous problem is considered for the case $\lambda_n \sim cn^{1/q}l(n)$, $c \neq 0, \infty$, $1 < q < 2$, where $l(n)$ is a slowly increasing function, and the corresponding theorem is proved, which makes it possible to find the limiting growth of the function $F(z)$ ensuring the existence of a nonzero Dirichlet series bounded on the real axis.

Let us introduce several auxiliary functions that we shall use below. Let $k(n) = n^{1/q}l_1(n)$, $1 < q < 2$; $l_1(n)$ is a slowly increasing function; then $\lambda_n \sim ck(n)$, $c \neq 0, \infty$. Denote by $\nu(r)$ the function inverse to $k(n)$, and by $h(x)$ the function conjugate in the sense of Young to the function $\nu(r)$ ($\tilde{h}(x) = \nu(x)$).

Definition. If the function $\psi(t)$ is defined for sufficiently large t and $\psi(t)/t \to +\infty$ as $t \to +\infty$, then the function

$$\tilde{\psi}(\xi) = \max_{t>0}(t\xi - \psi(t))$$

is called the function conjugate in the sense of Young to $\psi(t)$.

We formulate the main theorem in the case under consideration.

Theorem 1. Let there be a Dirichlet series

$$F(z)=\sum_{n=1}^{\infty}a_n e^{\lambda_n z},\qquad 0<\lambda_1<\lambda_2<\cdots,\tag{1}$$

absolutely convergent in the whole z-plane, and let

$$|F(x)|\leqslant C,\qquad -\infty<x<\infty.\tag{2}$$

Suppose that

$$\lim_{n\to\infty}\frac{\lambda_n}{k(n)}=c,\qquad c\neq 0,\infty.$$

If

$$M_F(x)\leqslant \mathrm{const}\cdot e^{(\alpha-\varepsilon)h(x)},\qquad \varepsilon>0,\quad h(x)=x^p l(x),\tag{3}$$

where

$$\alpha=\frac{c^p}{q^p}\frac{q-1}{(-\pi\,\mathrm{tg}\,\pi q/2)^{p-1}},\qquad \frac{1}{p}+\frac{1}{q}=1,$$

$h(x)$ is the function conjugate in the sense of ⋯

Putting c as the function $v(r)$, it follows from condition (2) that $F(z)\equiv 0$.

Accordingly, the following converse holds.

Theorem 2. If $\lambda_n\sim ck(n)$, $c\neq 0,\infty$, $1<q<2$, $l(n)$ is a slowly increasing function, and the condition

$$|\lambda_n^q-\lambda_k^q|\geq m|n-k|,\qquad m>0,$$

is satisfied, then there exists a function $Q(z)$, different from zero, representable by a Dirichlet series of the form (1), bounded on the real axis and satisfying the condition

$$M_Q(x)\leq \mathrm{const}\cdot e^{(a+\varepsilon)h(x)},\qquad \varepsilon>0,\quad h(x)=x^p l(x),\quad \frac{1}{p}+\frac{1}{q}=1.$$

The uniqueness theorem from note [(3)] is a special case for $k(n)=n^{1/q}$.

It is interesting to consider the case when an additional restriction is imposed on the sum of the Dirichlet series (1). Namely, if one assumes that condition (2) is fulfilled in the strip $|y| \leq \gamma_1$, $-\infty < x < \infty$, then the condition $|F(x+iy)| < ce^{-\delta|x|}$, $\delta < \lambda_1$, is fulfilled, and then for the function

$$\widetilde{F}(z) = \int_{-\infty}^{\infty} F(x)e^{-zx}\,dx,$$

regular in the domain $|\operatorname{Re} z| < \delta$, the estimate $|\widetilde{F}(-\sigma + iy)| < ce^{-\gamma_1|y|}$ will be valid. This estimate will not be reflected in the indicator of the function constructed in the proof of Theorem 1 in note [(3)], i.e., this restriction will not weaken the growth of the function $M_F(x)$.

Let us consider the case when the sum of the Dirichlet series (1) is regular in the entire plane and satisfies the condition

$$|F(x+iy)| \leq Me^{\beta|y|^p}. \tag{4}$$

Then the following holds.

Theorem 3. Let there be a Dirichlet series (1), absolutely convergent in the whole z-plane, and let condition (4) be satisfied. Suppose that

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{1/q}} = c, \qquad c \neq 0, \infty, \qquad 1 < q < 2.$$

If

$$M_F(x) \leq \text{const}\cdot e^{(\alpha-\varepsilon)x^p}, \qquad \varepsilon > 0,$$

where

$$\alpha = \frac{c^p}{q^p} \frac{q-1}{\left[\dfrac{c(\beta)}{\cos \pi q/2} c^q - \pi \operatorname{tg} \frac{\pi q}{2}\right]^{p-1}}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad c(\beta) = \frac{p-1}{\beta^{q-1} p^q},$$

then it follows from condition (4) that $F(z) \equiv 0$.

The methods of proof of the theorems formulated are analogous to the methods of proof in note [(3)].

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References

[1] M. A. Evgrafov, UMN, 17, no. 3 (105) (1962). [2] M. A. Evgrafov, *Asymptotic Estimates and Entire Functions*, Moscow, 1962. [3] G. M. Gashimov; DAN, 150, no. 4 (1963).

Note: Figure translations are in progress. See original paper for figures.