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Abstract

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ON UNITARY REPRESENTATIONS OF THE LORENTZ GROUP IN THE SPACE Π_k

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1. Let Π be a space with an indefinite metric and finite rank of indefiniteness k (see (1)), let G be a locally compact group, and let $g \to U_g$ be a unitary representation of the group G in the space Π_k; here unitarity is understood in the sense of the indefinite scalar product (ξ, η) in Π, so that $(U_g\xi, U_g\eta) = (\xi, \eta)$ for all $\xi, \eta \in \Pi_k$, $g \in G$. We shall say that the representation $g \to U_g$ of the group G **discretely contains** the representation $g \to V_g$, if in the space Π_k there exists a closed subspace $\mathfrak{M}$, invariant with respect to all operators U_g, and on which the restriction of the representation $g \to U_g$ is equivalent to the representation $g \to V_g$.

It is not difficult to show that (in contrast to ordinary unitary representations) every unitary representation in the space Π_k discretely contains an irreducible representation* (coinciding with the original representation if it itself is irreducible).

A representation $g \to W_g$ in a space R is called nonunitary if in R there is no nondegenerate scalar product (ξ, η) (definite or indefinite), invariant with respect to all operators of the representation. A unitary representation may discretely contain irreducible nonunitary representations. It turns out, however, that in the case of a complex semisimple Lie group the direct sum of all such representations is normally split off; thus the study of general unitary representations of a complex semisimple Lie group is reduced to the study of its unitary representations that do not discretely contain nonunitary representations.

For simplicity of exposition, we shall set out here (see item 4) a brief proof of this result for the case $G = \mathfrak{A}$, where $\mathfrak{A}$ is the complex unimodular group of second order (locally isomorphic to the Lorentz group). We first give a number of auxiliary propositions, some of which are of independent interest.

2. Let G be an arbitrary locally compact group and let $g \to U_g$ be a unitary representation of the group G in the space Π_k.

I**. If $\mathfrak{M}$ is a closed subspace in Π_k, invariant with respect to $g \to U_g$, and on which the restriction $g \to V_g$ of the representation $g \to U_g$ is irreducible and nonunitary, then $\mathfrak{M}$ is a null subspace in Π_k; consequently ***, $\dim \mathfrak{M} \leq k$, and the representation $g \to V_g$ is finite-dimensional.

We shall denote by $g \to \hat{V}_g$ the representation conjugate to $g \to V_g$, so that $\hat{V}_g = (V_g^{-1})'$, where A' denotes the operator adjoint to A.

II. Let the representation $g \to V_g$ in a finite-dimensional space R be the direct sum of two irreducible nonunitary representations $g \to V_g^{(1)}$ and $g \to V_g^{(2)}$ in the spaces R_1 and R_2, so that $R = R_1 \dotplus R_2$ and $V_g(\xi_1 + \xi_2) = V_g^{(1)}\xi_1 + V_g^{(2)}\xi_2$ for $\xi_1 \in R_1,\ \xi_2 \in R_2$. If in R one can introduce a nondegenerate scalar product in which the representation $g \to V_g$ is unitary, then the representations $g \to V_g^{(1)}$ and $g \to \hat{V}_g^{(2)}$ are equivalent.

III. An irreducible finite-dimensional representation $g \to V_g$ is unitary if and only if it is equivalent to the conjugate representation $g \to \hat{V}_g$.

From II and III we conclude:

IV. The direct sum of two irreducible nonunitary finite-dimensional representations equivalent to one another is a nonunitary representation.

We shall say that the representation $g \to V_g$ is contained p-fold in the representation $g \to U_g$ if: 1) in the space Π_k there exist closed subspaces $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ such that

$$U_g\xi_j = U_{1j}(g)\xi_j + U_{2j}(g)\xi_j + \dots + U_{lj}(g)\xi_l \quad \text{for } \xi_j \in \mathfrak{M}_j,\ j = 1, \dots, p,$$

where $U_{lj}(g)$, $l \le j$, is an operator from $\mathfrak{M}_j$ to $\mathfrak{M}_l$, and $g \to U_{jj}(g)$ is a representation of the group G in the space $\mathfrak{M}_j$, equivalent to the representation $g \to V_g$; 2) p is the maximal number satisfying condition 1). The subspace $\mathfrak{M} = \mathfrak{M}_1 + \dots + \mathfrak{M}_p$ will then be called the carrier of the representation $g \to V_g$ in the representation $g \to U_g$.

V. Every nonunitary irreducible representation $g \to V_g$ can be contained in a unitary representation $g \to U_g$ only with finite multiplicity, and the carrier $\mathfrak{M}$ of the representation $g \to V_g$ in the representation $g \to U_g$ is a null subspace; consequently, $\dim \mathfrak{M} \le k$.

VI. If in a unitary representation $g \to U_g$ two nonunitary irreducible representations $g \to V_g^{(1)}$ and $g \to V_g^{(2)}$ are contained with multiplicity, and if the representations $g \to V_g^{(1)}$ and $g \to \hat{V}_g^{(2)}$ are not equivalent, then the carriers of the representations $g \to V_g^{(1)}$ and $g \to V_g^{(2)}$ are mutually orthogonal.

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* This result was also obtained independently by R. S. Ismagilov.
** Propositions I-V partially overlap with results of Shlider (^{2}).
*** See (^{1}), Lemma 1.2.

Combining these propositions, we arrive at the following result:

Theorem 1. *A unitary representation in the space* Π_k *can discretely contain only a finite number of irreducible nonunitary representations and only with finite multiplicities, and the direct sum of the carriers of all these representations is a subspace of dimension* $\leq 2k$.

3. **Theorem 2.** *Let* $\mathcal{H}$ *be a family of permutable bounded Hermitian operators in the space* Π_k*, and let* ξ_0 *be a nonzero nonnegative vector from* Π_k *satisfying the conditions:*

1) $H\xi_0 = \lambda(H)\xi_0$ for all $H \in \mathcal{H}$, where $\lambda(H)$ is a numerical function on $\mathcal{H}$;

2) there exist $H \in \mathcal{H}$ such that $\lambda(H)$ is not real.

Then in Π_k *there exist two skew-conjugate null subspaces* $\mathfrak{N}, \mathfrak{N}'$*, and in them bases* $\xi_0, \xi_1, \dots, \xi_r$*;* $\xi'_0, \xi'_1, \dots, \xi'_r$*, possessing the following properties*

$$(\xi_\nu, \xi'_l) = \begin{cases} 0 & \text{for } \nu \neq l, \\ 1 & \text{for } \nu = l, \end{cases} \qquad \nu, l = 0, 1, \dots, r; \tag{1}$$

$$H\xi_0 = \lambda(H)\xi_0, \qquad H\xi_\nu = \lambda_{\nu 0}(H)\xi_0 + \dots + \lambda_{\nu,\nu-1}(H)\xi_{\nu-1} + \lambda(H)\xi_\nu,$$

$$\nu = 1, \dots, r; \tag{2}$$

$$H\xi'_r = \overline{\lambda(H)}\xi'_r, \qquad H\xi'_\nu = \overline{\lambda(H)}\xi'_\nu + \overline{\lambda_{\nu+1,\nu}(H)}\xi'_{\nu+1} + \dots + \overline{\lambda_{r,\nu}(H)}\xi'_r,$$

$$\nu = 0, 1, \dots, r-1, \tag{3}$$

for all $H \in \mathcal{H}$*, where* $\lambda_{\nu\rho}(H)$ *are numerical functions on* $\mathcal{H}$.

Remark 1. Let $\mathcal{H}$ be a family of permutable bounded Hermitian operators in Π_k, and suppose that one of the operators $H_0 \in \mathcal{H}$ has a nonreal eigenvalue λ_0. Then there exist a nonnegative vector $\xi_0 \neq 0$ and a numerical function $\lambda(H)$ on $\mathcal{H}$ such that $H\xi_0 = \lambda(H)\xi_0$ for all $H \in \mathcal{H}$, $\lambda(H_0) = \lambda_0$.

Remark 2. Theorem 2 and Remark 1 carry over, with the appropriate changes, to families of permutable unitary operators in Π_k.

4. We can now prove the following main theorem:

Theorem 3. *Let* $a \to U_a$ *be a unitary representation of the group* $\mathfrak{A}$ *in the space* Π_k*, and suppose that this representation discretely contains a nonunitary irreducible representation* $a \to V_a$*, acting in the invariant subspace* $\mathfrak{M}_0 \subset \Pi_k$.

Then:

1) *The representation* $a \to U_a$ *also discretely contains the conjugate representation* $a \to \hat{V}_a$*, acting in the invariant subspace* $\mathfrak{M}_0' \subset \Pi_k$*, skew-linked with* $\mathfrak{M}_0$*.*

2) *There exists a finite-dimensional subspace* $\Pi_{k'} \subset \Pi_k$*,* $k' \leq k$*, invariant with respect to the representation* $a \to U_a$ *and possessing the following properties: a) the restriction of the representation* $a \to U_a$ *to* $\Pi_k \ominus \Pi_{k'}$ *contains no discrete nonunitary irreducible representations; b) the restriction of the representation* $a \to U_a$ *to* $\Pi_{k'}$ *is an orthogonal sum of a finite number (possibly one) of representations, each of which is a direct sum of two mutually conjugate nonunitary representations acting in two skew-linked null subspaces.*

Proof. Consider the representations $x \to U_x$, $x \to V_x$ of the group ring ${}^*X$ of the group $\mathfrak{A}$, corresponding to the representations $a \to U_a$, $a \to V_a$. Let the indices j, q be chosen so that $\mathfrak{M}_j^q = E_{jj}^q \mathfrak{M}_0 \neq (0)$; then $\mathfrak{M}_j^q$ is one-dimensional. Let $\xi_0 \in \mathfrak{M}_j^q$, $\xi_0 \neq 0$; then

$$U_x \xi_0 = V_x \xi_0 = \lambda(x) \xi_0 \quad \text{for } x \in X_j^q, \tag{4}$$

where $\lambda(x)$ is a multiplicative linear functional on X_j^q. Since X_j^q is a commutative ring with involution, the operators U_x, $x = x^* \in X_j^q$, form a family $\mathcal{H}$ of commuting bounded Hermitian operators; moreover, because of the nonunitarity of the representation $a \to V_a$, the function $\lambda(x)$ takes non-real values for some $x = x^* \in X_j^q$. Consequently, Theorem 2 is applicable to this family $\mathcal{H}$. Let $\xi_0, \dots, \xi_r$ and $\xi_0', \dots, \xi_r'$ be basis vectors satisfying conditions (1)–(3); put $\mathfrak{M}_\nu = \{U_x \xi_\nu, \ x \in X\}$, $\mathfrak{M}_\nu' = \{U_x \xi_\nu', \ x \in X\}$, $\nu = 0, 1, \dots, r$. It is easy to show that $E_{jj}^q \xi_\nu = \xi_\nu$, $E_{jj}^q \xi_\nu' = \xi_\nu'$. Hence, for $x, y \in X$,

$$(U_x \xi_\nu, U_y \xi_\nu) = (U_{y^* x} \xi_\nu, \xi_\nu) = (U_{y^* x} E_{jj}^q \xi_\nu, E_{jj}^q \xi_\nu) =$$

$$= (U_{x'} \xi_\nu, \xi_\nu) = \lambda(x')(\xi_\nu, \xi_\nu) = 0,$$

where $x' = e_{jj}^q y^* x e_{ij}^q \in X_j^q$ (recall that $(\xi_\nu, \xi_\nu) = 0$); consequently, $\mathfrak{M}_\nu$ is a null subspace and, analogously, $\mathfrak{M}_\nu'$ is a null subspace. Therefore $\dim \mathfrak{M}_\nu \leq k$, $\dim \mathfrak{M}_\nu' \leq k$. On the other hand, every irreducible finite-dimensional representation of the group $\mathfrak{A}$ is completely reducible. Using this fact, the vectors $\xi_0, \dots, \xi_r$ and $\xi_0', \dots, \xi_r'$ may be chosen so that: α) $U_x \xi_\nu = \lambda(x) \xi_\nu$, $U_x \xi_\nu' = \overline{\lambda}(x) \xi_\nu'$ for $\nu = 0, 1, \dots, r$; β) the restriction of the representation $a \to U_a$ to $\mathfrak{M}_\nu$ and to $\mathfrak{M}_\nu'$ is irreducible. By α), on $\mathfrak{M}_\nu$ it is equivalent to the representation $a \to V_a$, and on $\mathfrak{M}_\nu'$ to the representation $a \to \hat{V}_a$. Let us consider, in particular, $\mathfrak{M}_0$ and $\mathfrak{M}_0'$; they are skew-linked. Indeed, the set $\mathfrak{N}_0$ of all ξ in $\mathfrak{M}_0$ orthogonal to $\mathfrak{M}_0'$ forms a subspace in $\mathfrak{M}_0$ invariant with respect to all V_a, different from $\mathfrak{M}_0$, since $(\xi_0, \xi_0') = 1$; consequently, by irreducibility of the representation $a \to V_a$, $\mathfrak{N}_0 = (0)$. Similarly $\mathfrak{N}_0' = (0)$, where $\mathfrak{N}_0'$ is the set of all ξ' in $\mathfrak{M}_0'$ orthogonal to

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* Here and below we use the notation and terminology of [3], chiefly § 15 in [3]; the operator E_{ii}^{q} corresponds to the representation $a \to U_a$.

to $\mathfrak{M}_0$, and this means that $\mathfrak{M}_0$ and $\mathfrak{M}_0'$ are skew-associated. Consequently, for $\mathfrak{M}_0$ and $\mathfrak{M}_0'$ assertion 1) of Theorem 3 holds.

Let us now prove assertion 2). $\mathfrak{M}_0 \dotplus \mathfrak{M}_0'$ is a finite-dimensional space Π_{k_1}, $k_1 \leq k$, invariant with respect to all U_a, so that*

$$\Pi_k = (\mathfrak{M}_0 \dotplus \mathfrak{M}_0') \oplus \Pi_{k-k_1},$$

where

$$\Pi_{k-k_1} = (\mathfrak{M}_0 \dotplus \mathfrak{M}_0')^{\perp}$$

is also invariant with respect to all U_a. If the representation $a \to U_a$ does not contain discrete nonunitary representations in Π_{k-k_1}, then for $\Pi_{k'} = \Pi_{k_1}$ assertion 2) of the theorem holds. Otherwise, applying the preceding argument to the restriction of the representation $a \to U_a$ to Π_{k-k_1}, we obtain that

$$\Pi_{k-k_1} = (\mathfrak{M}_0^{(1)} \dotplus \mathfrak{M}_0^{(1)'}) \oplus \Pi_{k-k_1-k_2},$$

where $\mathfrak{M}_0^{(1)}$, $\mathfrak{M}_0^{(1)'}$ are skew-associated, and on them the restrictions of the representation $a \to U_a$ are irreducible, nonunitary, and mutually conjugate. Repeating this argument and taking into account that $k_1 + k_2 + \cdots \leq k$, after a finite number of steps we obtain a subspace of the form

$$\Pi_{k'} = (\mathfrak{M}_0 \dotplus \mathfrak{M}_0') \oplus \cdots \oplus (\mathfrak{M}_0^{(\nu)} \dotplus \mathfrak{M}_0^{(\nu)'}),$$

possessing properties a) and b).

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* Π_0 denotes the ordinary Hilbert space.