Aerodynamics

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1963

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Abstract

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A Method for Computing Rarefied-Gas Flows

(Presented by Academician L. I. Sedov, March 1, 1963)

At present, formulations of problems on the motion of a rarefied gas are known, for example, using integral kinetic equations ([1]). The main difficulty in solving problems in the indicated formulation is the need to carry out computations with functions of many variables (four or more); computations of this kind lead to substantial difficulties even when modern computing machines are used. To overcome this difficulty it is desirable to pass to functions of a smaller number of variables, even at the cost of increasing the number of unknown functions.

In the present work a way is indicated for constructing, starting from the integral equation

$$f = V(f), \tag{1}$$

systems of equations for many functions of a small number of independent variables, the determination of which is sufficient for an approximate solution of problems of rarefied-gas aerodynamics. In the special case when the gas, whose molecular model is an absolutely elastic smooth sphere, occupies all space and there are no bodies immersed in it, such a construction is carried through to the end. A system of equations is written out for one-dimensional nonstationary problems.

1. We start from the representation, already used by Grad ([2]), of the distribution function in terms of Hermite polynomials of a three-dimensional argument

$$f(\mathbf{x}, \mathbf{u}, t) = N(2\pi RT)^{-3/2} \exp\left\{-\frac{v^2}{2}\right\} \sum_{n=0}^{\infty} \frac{a^{(n)}(\mathbf{x}, t)}{n!} \mathcal{H}^{(n)}(\mathbf{v}), \tag{2}$$

where $\mathbf{u} = \vec{\xi} + \mathbf{v}\sqrt{RT}$, and

$$N = N(\mathbf{x}, t) = \iiint_{\infty} f(\mathbf{x}, \mathbf{u}, t)\, d\mathbf{u}; \tag{3}$$

$$\vec{\xi} = \vec{\xi}(\mathbf{x}, t) = \frac{1}{N} \iiint_{\infty} \mathbf{u} f(\mathbf{x}, \mathbf{u}, t)\, d\mathbf{u}; \tag{4}$$

$$T = T(\mathbf{x}, t) = \frac{1}{3N} \iiint_\infty |\mathbf{u} - \vec{\xi}|^2 f(\mathbf{x}, \mathbf{u}, t)\, d\mathbf{u}; \tag{5}$$

R is the gas constant; the quantities $a^{(n)}$ are determined from the relations

$$a^{(n)}(\mathbf{x}, t) = N^{-1}(RT)^{3/2} \iiint_\infty \mathcal{H}^{(n)}(\mathbf{v}) f(\mathbf{x}, \vec{\xi} + \mathbf{v}\sqrt{RT}, t)\, d\mathbf{v}. \tag{6}$$

Under the conditions indicated above, the operator $V(f)$ has the form (here σ is the molecular-collision cross section)

$$\begin{aligned} V(f) = \int_{-\infty}^{t} \frac{1}{2} \int_\infty\int_\infty\int_\infty\int_\infty\int_\infty\int_\infty |\mathbf{u}_1 - \mathbf{u}_2|\, \sigma T(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2)\, f[\mathbf{x} - \mathbf{u}(t-\tau), \mathbf{u}_1, \tau] \times \\ \times f[\mathbf{x} - \mathbf{u}(t-\tau), \mathbf{u}_2, \tau] \times \\ \times \exp\left\{ -\int_{\tau}^{t} \int_\infty\int_\infty\int_\infty |\mathbf{u} - \mathbf{u}_3|\, \sigma f[\mathbf{x} - \mathbf{u}(t-q), \mathbf{u}_3, q]\, d\mathbf{u}_3\, dq \right\} d\mathbf{u}_1\, d\mathbf{u}_2\, d\tau \end{aligned} \tag{7}$$

and contains 11 quadratures, of which only 2 are taken in physical space. If the expansion (2) is substituted into (7), then 9 quadratures in velocity space are evaluated analytically.

We have:

$$Q(\mathbf{x}, \mathbf{u}, t) = \int_\infty\int_\infty\int_\infty |\mathbf{u} - \mathbf{u}_1|\, \sigma f(\mathbf{x}, \mathbf{u}_1, t)\, d\mathbf{u}_1 =$$

$$= \sigma N \sqrt{2RT} \sum_{n=0}^{\infty} \frac{a^{(n)}(\mathbf{x}, t)}{n!} \sum_{z=0}^{[\frac{n}{2}]} \vec{\alpha}^{\,n-2z} \vec{\delta}^{\,z} E_{nz} \left(\left| \frac{\mathbf{u} - \vec{\xi}}{\sqrt{2RT}} \right| \right); \tag{8}$$

$$\begin{aligned} \Phi(\mathbf{x}, \mathbf{u}, t) = \frac{1}{2} \int_\infty\int_\infty\int_\infty\int_\infty\int_\infty\int_\infty |\mathbf{u}_1 - \mathbf{u}_2|\, \sigma T(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2) \times \\ \times f(\mathbf{x}, \mathbf{u}_1, t) f(\mathbf{x}, \mathbf{u}_2, t)\, d\mathbf{u}_1\, d\mathbf{u}_2 = \end{aligned}$$

$$= (2\pi)^{-3/2} \frac{\sigma N^2}{RT} \exp\left\{ -\frac{|\mathbf{u} - \vec{\xi}|^2}{2RT} \right\} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^{(m)} a^{(n)}}{m!\, n!} \times$$

$$\times \sum_{z=0}^{[\frac{m+n}{2}]} \vec{\alpha}^{\,m+n-2z} \vec{\delta}^{\,z} F_{mnz} \left(\left| \frac{\mathbf{u} - \vec{\xi}}{\sqrt{2RT}} \right| \right). \tag{9}$$

Here: $\vec{\alpha} = \mathbf{v}/v$ is a unit vector; $\vec{\delta} = \{\delta_{ij}\}$ is the unit tensor of rank two;

$$E_{nz}\left(\frac{x}{\sqrt{2}}\right) = (2\pi)^{-3/2}\sum_{k=0}^{\infty}(-1)^{z+k-1}\frac{x^{n-2z+2k}}{2^{k-1}k![\,4(n-z+k)^2-1\,]}; \qquad (10)$$

$$F_{mnz}\left(\frac{x}{\sqrt{2}}\right) = (2\pi)^{-3/2}(-1)^{m+z}\sum_{k=0}^{\min\{[\frac{m}{2}],z\}} C_z^k \times$$
$$\times \sum_{i=0}^{\min\{m+n-2z,\, m-2k\}} (-1)^i C^i_{m+n-2z} C^n_{m+n-2k-i} \times$$
$$\times \sum_{\mu=0}^{[\frac{m+n-i}{2}]-z} \frac{(2\mu-1)!!}{(2z-2k+2\mu)!!} C^{2\mu}_{m+n-2z-i} \sum_{\nu=0}^{z-k+\mu} (-1)^\nu \times$$
$$\times C^\nu_{z-k+\mu} \sum_{s=0}^{[\frac{n}{2}]} (-1)^s (2s-1)!!\, C_n^{2s} x^{-m-n+2k+2i+2\nu-1} \times$$
$$\times \left\{ \sum_{q=0}^{[\frac{m-i+1}{2}]-k} \frac{(2q-1)!!}{m+n-2k-s-i-\nu-q+1} C^{2q}_{m-2k-i+1} \times \right.$$
$$\times \left[\left(x^{2m+2n-4k-2s-2i-2\nu-2q+2} - (2m+2n-4k-2s-2i-2\nu-2q+1)!! \right) \times \right.$$
$$\times \sqrt{\frac{\pi}{2}}\ \mathrm{erf}\,\frac{x}{\sqrt{2}} + \sum_{j=0}^{m+n-2k-s-i-\nu-q} \frac{(2m+2n-4k-2s-2i-2\nu-2q+1)!!}{(2m+2n-4k-2s-2i-2\nu-2q-2j+1)!!} \times$$
$$\left. \times x^{2(m+n-2k-s-i-\nu-q-j)+1} \exp\left\{-\frac{x^2}{2}\right\} \right] + 2 \sum_{q=0}^{[\frac{m-i}{2}]-k} \sum_{\eta=0}^{m-2k-i-2q} (-1)^\eta \frac{(2q+\eta)!!}{\eta!!} \times$$
$$\times C^{2q+\eta+1}_{m-2k-i+1} \left[(2m+2n-4k-2s-2i-2\nu-2q-1)!! \sqrt{\frac{\pi}{2}}\ \mathrm{erf}\,\frac{x}{\sqrt{2}} - \right.$$
$$- \sum_{j=0}^{m+n-2k-s-i-\nu-q-1} \frac{(2m+2n-4k-2s-2i-2\nu-2q-1)!!}{(2m+2n-4k-2s-2i-2\nu-2q-2j-1)!!} \times$$
$$\left.\left. \times x^{2(m+n-2k-s-i-\nu-q-j)-1} \exp\left\{-\frac{x^2}{2}\right\} \right] \right\}. \qquad (11)$$

Using expressions (10) and (11), we have compiled tables of the functions $E_{nz}(x)$ (for $0 \le n \le 5$, $0 \le z \le n/2$) and $F_{mnz}(x)$ (for $0 \le m+n \le 5$, $0 \le z \le (m+n)/2$) for $0 \le x \le 4.5$ with step 0.25.

Substitution of (8) and (9) into (7) and into (1), and then into (6) and into (3), (4), (5), gives an infinite system of nonlinear integral equations for the functions

$N,\ \vec{\xi},\ T,\ a^{(n)}$, depending only on the variables x and t. This system can be solved by truncating the expansion (2).

II. In the case of one-dimensional nonstationary flows, the finite system of equations obtained in this way contains a finite number of functions of two independent variables. As an example, we give this system for the case in which terms with $n > 2$ are discarded.

$$N = 2\pi T^{3/2} \int_{-\infty}^{\infty} dv_x \int_0^{\infty} v_y\, dv_y \int_{-\infty}^{t} \Phi_\tau \exp\left\{-\int_\tau^t Q_q\, dq\right\} d\tau,$$

$$0 = \int_{-\infty}^{\infty} v_x\, dv_x \int_0^{\infty} v_y\, dv_y \int_{-\infty}^{t} \Phi_\tau \exp\left\{-\int_\tau^t Q_q\, dq\right\} d\tau,$$

$$N a_{xx} = 2\pi T^{3/2} \int_{-\infty}^{\infty} (v_x^2 - 1)\, dv_x \int_0^{\infty} v_y\, dv_y \int_{-\infty}^{t} \Phi_\tau \exp\left\{-\int_\tau^t Q_q\, dq\right\} d\tau,$$

$$N a_{yy} = 2\pi T^{3/2} \int_{-\infty}^{\infty} dv_x \int_0^{\infty} \left(\frac{1}{2} v_y^3 - v_y\right) dv_y \int_{-\infty}^{t} \Phi_\tau \exp\left\{-\int_\tau^t Q_q\, dq\right\} d\tau, \tag{12}$$

$$a_{xx} + 2a_{yy} = 0,$$

$$a_{zz} = a_{yy}.$$

Here:

$$\Phi_\tau = \pi^{-3/2} \frac{\sigma}{2R} N_\tau^2 T_\tau^{-1} \exp\left\{-\frac{1}{2} v_\tau^2\right\} \times$$

$$\times \left[F_{000}\left(\frac{1}{\sqrt{2}} v_\tau\right) + \left(a_{xx\tau}\alpha_{x\tau}^2 + a_{yy\tau}\alpha_{y\tau}^2\right) F_{200}\left(\frac{1}{\sqrt{2}} v_\tau\right)\right],$$

$$Q_\tau = \sigma N_\tau \sqrt{R T_\tau} \left[E_{00}\left(\frac{1}{\sqrt{2}} v_\tau\right) - \frac{1}{2}\left(a_{xx\tau}\alpha_{x\tau}^2 + a_{yy\tau}\alpha_{y\tau}^2\right) E_{20}\left(\frac{1}{\sqrt{2}} v_\tau\right)\right],$$

$$N_\tau = N(x_\tau, \tau), \qquad \vec{\xi}_\tau = \vec{\xi}(x_\tau, \tau), \qquad T_\tau = T(x_\tau, \tau),$$

$$a_{xx\tau} = a_{xx}(x_\tau, \tau), \qquad a_{yy\tau} = a_{yy}(x_\tau, \tau), \qquad x_\tau = x - (t-\tau)\left(v_x\sqrt{RT_t} + \xi_{xt}\right),$$

$$\mathbf{v}_\tau = \frac{1}{\sqrt{RT_\tau}}\left(\mathbf{v}\sqrt{RT_t} + \vec{\xi}_t - \vec{\xi}_\tau\right),$$

$$v_\tau = \frac{1}{\sqrt{RT_\tau}}\sqrt{\left(v_x\sqrt{RT_t} + \xi_{xt} - \xi_{x\tau}\right)^2 + v_y^2 RT_t},$$

$$\alpha_{x\tau} = \frac{v_{x\tau}}{v_\tau}, \qquad \alpha_{y\tau} = \frac{v_{y\tau}}{v_\tau}.$$

The solution of systems of the form (12), also written out for $n > 2$, can be carried out on machines.

In conclusion, I consider it my pleasant duty to express my gratitude to R. G. Barantsev for information about the investigations he is conducting independently in the same direction [3], which stimulated the writing of this note, and to Prof. S. V. Vallander for the great assistance rendered in its preparation.

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Received
27 II 1963

REFERENCES

[1] S. V. Vallander, DAN, **131**, No. 1 (1960).
[2] H. Grad, Comm. Pure and Appl. Math., **2**, No. 4, 325 (1949).
[3] R. G. Barantsev, DAN, **151**, No. 5 (1963).

Note: Figure translations are in progress. See original paper for figures.