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COUNTABILITY CONDITIONS IN LOCALLY CONVEX SPACES

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Abstract

Full Text

COUNTABILITY CONDITIONS IN LOCALLY CONVEX SPACES

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Djedonne, in paper ⁽²⁾, proved that if in a barrelled or bornological space E there exists a fundamental sequence of bicomact sets, then E is a dense subspace of the strong dual E_1 to a Fréchet–Montel space. In this article Djedonne posed two questions:

Does E always coincide with E_1 ?

Will the stated result be true for infrabarrelled spaces?

In § 1 an affirmative answer is given to Djedonne's questions. In § 2 locally convex spaces with a fundamental sequence of bounded sets are studied; it turns out that for these spaces the result of Djedonne stated above is true.

§ 1. A fundamental sequence of sets of a certain class \mathfrak{A} is a sequence of sets of this class such that every set from \mathfrak{A} is contained in some set of this sequence. A fundamental sequence of bicomact sets in a topological vector space will, for brevity, be called a K -sequence.

Lemma 1. *If in a topological vector space E there exists a K -sequence, then every bounded set in E is relatively bicomact (i.e. has bicomact closure).*

Proof. Let A be a bounded but not relatively bicomact set, and let $(K)_{n \in \mathbb{N}}$ be a K -sequence in E . Then, whatever n may be, $A \not\subset nK_n$ (since $x \rightarrow nx$ is a homeomorphism of E onto E , and therefore nK_n is bicomact). Consequently, for each n there exists $x_n \in A$ such that $x_n \notin nK_n$. Since A is bounded, $x_n/n \rightarrow 0$, and therefore the set $K = \{x_n/n, 0\}_{n \in \mathbb{N}}$ is bicomact. By assumption, then $K \subset K_p$ for some p . But this is impossible, since $x_p \notin pK_p$, whence $x_p/p \notin K_p$.

A locally convex space is called **infrabarrelled***, if in it every convex set absorbing all bounded sets is a neighborhood of zero. All barrelled and all bornological spaces are infrabarrelled.

Theorem 1. *An infrabarrelled space E in which there exists a K -sequence is the strong dual to a Fréchet–Montel space.*

Proof. By the lemma, the topology of compact convergence T_k on E' coincides with the strong topology $\beta(E', E) = \beta$, and, moreover, $(E', \beta) = (E', T_k)$ is metrizable. Since E is infrabarrelled, every bounded set of (E', β) is equicontinuous and therefore (since $T_k = \beta$) relatively bicomact. Consequently, (E', β) is quasi-complete and hence, by virtue of its metrizability, complete. Thus (E', β)

is a complete metrizable space with relatively bicomact bounded sets, i.e. a Fréchet–Montel space. Thus

* ⁽¹⁾, Ex. 12 to § 2, Ch. III. Concerning the properties used of the strong dual to an infrabarrelled space, see ⁽¹⁾, Ex. 6 to § 3, Ch. IV.

as in E every bounded set is relatively bicomact (Lemma 1), E is semireflexive, i.e. $(E', \beta)' = E$. It remains to note that since E is infrabarrelled, its topology coincides with $\beta(E, E')$. Thus, E is the strong dual of the Fréchet–Montel space (E', β) .

Corollary. *Every infrabarrelled space in which there exists a K -sequence is complete.*

Proof. The strong dual of a Fréchet space is complete.

§ 2. For brevity, a fundamental sequence of closed bounded sets in a topological vector space will be called a P -sequence.

Lemma 1'. *If in a topological vector space E there exists a P -sequence, then in E every bounded set is closed bounded.*

The proof is analogous to the proof of Lemma 1.

Lemma 2. *If E is an infrabarrelled space in which there exists a P -sequence, then every bounded set from its completion \hat{E} is contained in the completion of some closed bounded set from E .*

Proof. By Lemma 1', every bounded set from E is closed bounded. As in the proof of Theorem 1, one can show that (E', T_p) , where T_p is the topology of uniform convergence on closed bounded sets, is a Fréchet–Montel space and, consequently, is both barrelled and boundedly closed simultaneously. From the latter it follows that E is regular and $\hat{E} \subseteq E''^*$. Let A be a bounded set in \hat{E} ; then A is bounded in $(\hat{E}, \sigma(\hat{E}, E'))$, and therefore also in $(E'', \sigma(E'', E'))$. By the regularity of E , there exists a bounded set $K \subset E$, $\sigma(E'', E')$ -dense in A . Since $K \subset \hat{E}$ and the topology $\sigma(\hat{E}, E')$ is induced on \hat{E} by the topology $\sigma(E'', E')$, it follows that $\overline{K}^{(\hat{E}, \sigma(\hat{E}, E'))}$ (the $\sigma(\hat{E}, E')$ -closure of K in \hat{E}) contains A . But $\overline{K}^{(\hat{E}, \sigma(\hat{E}, E'))} = \overline{K}^{\hat{E}}$, and $\overline{K}^{\hat{E}} = \overline{K}^E$; hence $\hat{K} \supset A$. It remains to add that, by Lemma 1', K is closed bounded in E .

Remark. If in an infrabarrelled space E there exists a P -sequence, then in \hat{E} there exists a K -sequence. Indeed, it follows from Lemma 2 that if $(P)_{n \in \mathbb{N}}$ is a P -sequence in E , then (\hat{P}_n) is a K -sequence in \hat{E} .

Theorem 2. *If E is an infrabarrelled space in which there exists a P -sequence, then \hat{E} is the strong dual of a Fréchet–Montel space.*

Proof. Since E is infrabarrelled, \hat{E} is barrelled. At the same time, by the remark to Lemma 2, in \hat{E} there exists a K -sequence, so it remains to apply Theorem 1.

In conclusion we shall show that in the space c_0 of numerical null sequences, endowed with the topology T induced from $(m, \tau(m, l^1))$, where $\tau(m, l^1)$ is the Mackey topology in the space m of bounded sequences, there is a P -sequence, but there is no K -sequence. Indeed, in l^1 every weakly bicomact set is strongly bicomact, so that on every equicontinuous set from m the topologies $\tau(m, l^1)$ and $\sigma(m, l^1)$ coincide. Consequently, the balls

$$B_n = \{(x_i) \in m : \sup_{i \in N} |x_i| \leq n\}$$

in m , being equicontinuous sets, are bicomact in $(m, \tau(m, l^1))$. Since they form a fundamental sequence of bounded sets in $(m, \beta(m, l^1))$, and every bicomact set in $(m, \tau(m, l^1))$ is bounded in

* See (1), exercise 10 to § 3 of Chapter IV. An infrabarreled space is called **regular** if every bounded set from $(E'', \sigma(E'', E'))$ is contained in the closure (in the topology $\sigma(E'', E')$) of some bounded set from E .

$(m, \beta(m, l^1))$, the balls B_n form a K -sequence in $(m, \tau(m, l^1))$. But B_n is the closure in $(m, \tau(m, l^1))$ of the ball

$$A_n = \{(x_i) \in c_0 : \sup_{i \in N} |x_i| \leq n\}$$

from c_0 , and, since $(m, \tau(m, l^1))$ is complete, $(\hat{c}_0, \hat{T}) = (m, \tau(m, l^1))$. Thus the balls A_n in (c_0, T) are quite bounded and form a P -sequence. If there were now a K -sequence in (c_0, T) , then (by Lemma 1) the balls A_n in (c_0, T) would be relatively bicomact, and then their closures in (\hat{c}_0, \hat{T}) would be complete; but the completion of the ball A_n in the topology T is the ball B_n , and it does not belong to c_0 .

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CITED LITERATURE

¹ N. Bourbaki, *Topological Vector Spaces*, II, 1958. ² T. Dieudonné, Proc. Am. Math. Soc., 8, 367 (1957).

Note: Figure translations are in progress. See original paper for figures.

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