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Abstract

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ON TRANSFERENCE THEOREMS

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Let $\bar{x} = \max(1, |x|)$, and let $\langle x \rangle$ denote the distance from x to the nearest integer. Consider the linear forms

$$L_j(x_1, \dots, x_m) = \sum_{i=1}^{m} a_{ij} x_i; \qquad M_i(u_1, \dots, u_n) = \sum_{j=1}^{n} a_{ij} u_j.$$

From the classical transference theorems of A. Ya. Khinchin ([1,2]) it follows that, if there is no nontrivial integral solution of the inequalities

$$\max \langle M_i(u_1, \dots, u_n) \rangle \leq D; \qquad \max |u_j| \leq Y \quad (i = 1, \dots, m;\ j = 1, \dots, n), \quad (1)$$

then there is also no nontrivial integral solution for the inequalities

$$\max \langle L_j(x_1, \dots, x_m) \rangle \leq C; \qquad \max |x_i| \leq X \quad (i = 1, \dots, m;\ j = 1, \dots, n),$$

where

$$X = C_1(m, n) \frac{Y^n}{D^{1-m}}; \qquad C = C_2(m, n) \frac{D^m}{Y^{1-n}}$$

(see, for example, Theorem 2, Ch. 5 in ([3])).

Transference theorems in which, as above, we deal with maxima of linear forms and variables will be called **cubic**. We shall consider transference theorems of another kind, where conditions are imposed on products of linear forms and variables. In particular, conditions (1) are replaced by conditions of the form

$$(\bar{x}_1 \dots \bar{x}_n)^\gamma [\ln(\bar{x}_1 + 1) \dots \ln(\bar{x}_n + 1)]^\beta \prod_{i=1}^{m} \langle M_i(x_1, \dots, x_n) \rangle \geq C_0.$$

We shall call these the **hyperbolic** transference theorems. A special case of theorems of this type was considered by A. Ya. Khinchin ([2]) and later by

Dyson ([4]). Usually the proofs of various transference theorems are based on the application of Dirichlet' s principle (an exception is the analytic theorem of A. O. Gelfond ([3])). In the present paper, in contrast to this, another method of proof is used, proposed by N. M. Korobov. It is based on the estimation of trigonometric sums.

§ 1. Consider the inequalities

$$\langle a_1 x_1 + \dots + a_n x_n \rangle \geq \frac{C}{\bar{x}_1 \dots \bar{x}_n \ln^{\gamma}(\bar{x}_1 + 1) \dots \ln^{\gamma}(\bar{x}_n + 1)}, \tag{2}$$

$$\langle a_1 x \rangle \cdots \langle a_{nx} \rangle \geq \frac{C_1}{x \ln^{\gamma_1}(x+1)}, \tag{3}$$

where $C = C(a_i, n)$, $C_1 = C_1(a_i, n)$.

Theorem 1. If inequality (2) is satisfied for any integers $(x_1, \dots, x_n) \neq (0, \dots, 0)$, then for every integer $x > 0$, with $\gamma_1 = (\gamma + 2)n^2$, inequality (3) is also satisfied. The converse theorem is also true with $\gamma = (\gamma_1 + 1 + n)n$.

We shall confine ourselves to proving only the second part of the theorem. The proof is based on the following

Lemma. If for $m > 0$ the inequality

$$\langle\langle \alpha_1 m \rangle\rangle \cdots \langle\langle \alpha_s \rangle\rangle \geq \frac{C_2}{m \ln^{\gamma}(m+1)},$$

holds, then

$$\sum_{k=1}^{m} \frac{1}{\langle\langle \alpha_1 k \rangle\rangle \cdots \langle\langle \alpha_s k \rangle\rangle} \leq C_3 m \ln^{\gamma+s}(m+1).$$

This assertion is obtained with the aid of Abel' s transformation and the application of Dirichlet' s principle.

Let us pass to the proof of the theorem. Denote by $N_{p_1,\dots,p_n}(0,\alpha)$ the number of incidences of $\{\alpha_1 x_1 + \dots + \alpha_n x_n\}$ in $(0,\alpha)$ for $1 \leq x_1 \leq p_1, \dots, 1 \leq x_n \leq p_n$. Introduce for the interval $(0,\alpha)$ functions $\psi(x)$ and $\psi_1(x)$, making it possible to approximate $N_{p_1,\dots,p_n}(0,\alpha)$ from above and below (see ([5]), p. 260). Then

$$\sum_{x_1=1}^{p_1} \cdots \sum_{x_n=1}^{p_n} \psi_1(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq N_{p_1,\dots,p_n}(0,\alpha) \leq$$

$$\leq \sum_{x_1=1}^{p_1} \cdots \sum_{x_n=1}^{p_n} \psi(\alpha_1 x_1 + \dots + \alpha_n x_n).$$

Hence we obtain that

$$R = \left|N_{p_1,\ldots,p_n}(0,\alpha) - p_1 \cdots p_n \alpha\right| \leq$$

$$\leq 2\left|\sum_{m=1}^{\infty} C(m) \sum_{x_1,\ldots,x_n} \exp\left[2\pi i \left(\sum_{i=1}^{n} \alpha_i x_i\right) m\right]\right| + 2p_1 \cdots p_n \delta,$$

where

$$|C(m)| < \begin{cases} \dfrac{1}{\pi m}, & \text{for } \delta < \dfrac{1}{\pi m}, \\ \dfrac{1}{\pi^2 m^2 \delta}, & \text{for } \delta \geq \dfrac{1}{\pi m}. \end{cases}$$

Choose

$$\delta = \frac{r_1}{\pi p_1 \cdots p_n}$$

and split the interval of summation over m

$$R \leq C_4 \left(\sum_{m=1}^{p_1 \cdots p_n} \frac{1}{m\langle\langle \alpha_1 m\rangle\rangle \cdots \langle\langle \alpha_{nm}\rangle\rangle} + \sum_{m=p_1 \cdots p_n+1}^{\infty} \frac{p_1 \cdots p_n}{m^2 \langle\langle \alpha_1 m\rangle\rangle \cdots \langle\langle \alpha_{nm}\rangle\rangle}\right).$$

Using Abel’ s transformation and applying the lemma, we obtain for the first sum

$$|S_1| \leq C_5 \left(\frac{1}{p_1 \cdots p_n} \sum_{m=1}^{p_1 \cdots p_n} \frac{1}{\langle\langle \alpha_1 m\rangle\rangle \cdots \langle\langle \alpha_{nm}\rangle\rangle} + \sum_{m=1}^{p_1 \cdots p_n} \frac{1}{m^2} \sum_{k=1}^{m} \frac{1}{\langle\langle \alpha_1 k\rangle\rangle \cdots \langle\langle \alpha_{nk}\rangle\rangle}\right) \leq$$

$$\leq C_6 \left[\ln^{\gamma_1+n}(p_1 \cdots p_n + 1) + \sum_{m=1}^{p_1 \cdots p_n} \frac{\ln^{\gamma_1+n}(m+1)}{m}\right] \leq$$

$$\leq 2C_6 \ln^{\gamma_1+n+1}(p_1 \cdots p_n + 1).$$

Similarly,

$$|S_2| \leq C_7 \ln^{\gamma_1+n}(p_1 \cdots p_n + 1),$$

$$R \le C_8 \ln^{\gamma_1+n+1}(p_1 \cdots p_n + 1). \tag{4}$$

We shall show that if estimate (4) holds for $p_1, \dots, p_n$, then for $x_i = \pm[p_i/R]$ the inequality

$$\{a_1x_1 + \cdots + a_nx_n\} \le \left\langle \frac{k}{4p_1 \cdots p_n} \right\rangle = \frac{C_9}{R^n \bar{x}_1 \cdots \bar{x}_n} \tag{5}$$

will follow.

Indeed, suppose

$$\{a_1x_1 + \cdots + a_nx_n\} \le \frac{1}{4p_1 \cdots p_n}.$$

Choose $T = 2R$, $\alpha = R/p_1 \cdots p_n$. Then, evidently,

$$\{k(a_1x_1 + \cdots + a_nx_n)\} \le \frac{k}{4p_1 \cdots p_n} < \alpha \quad \text{for } k = 1, \dots, T.$$

Thus, on the interval $(0, \alpha)$ there lie at least T fractional parts. Consequently,

$$T \le N_{p_1,\dots,p_n}(0, \alpha) < \alpha p_1 \cdots p_n + R = 2R = T.$$

The contradiction obtained proves inequality (5). It is easy to see that from (5) the estimate

$$\{a_1x_1 + \cdots + a_nx_n\} \ge \frac{C_{10}}{\bar{x}_1 \cdots \bar{x}_n \ln^{(v_1+1+n)n}(\bar{x}_1 \cdots \bar{x}_n + 1)} \tag{6}$$

follows, where, as above, $x_i = \pm[p_i/R]$.

When $p_1, \dots, p_n$ run through all natural numbers, the collection $x_1, \dots, x_n$ assumes all integer values. By the same method one obtains the inequality

$$1 - \{a_1x_1 + \cdots + a_nx_n\} \ge \frac{C_{10}}{\bar{x}_1 \cdots \bar{x}_n \ln^{(v_1+1+n)n}(\bar{x}_1 \cdots \bar{x}_n + 1)}. \tag{7}$$

From (6) and (7) the assertion of the theorem follows.

Remark 1. Analogously one obtains transference theorems relating a system of linear forms to the transposed system.

Remark 2. With the aid of A. O. Gel' fond' s theorem ([3]), similar theorems are obtained, but with less precise estimates.

2. Let p be an arbitrary integer greater than zero. Consider the systems of congruences

$$\left.\begin{aligned} a_{11}x_1+\dots+a_{1n}x_n &\equiv x_{n+1},\\ &\dots\\ a_{m1}x_1+\dots+a_{mn}x_n &\equiv x_{n+m}, \end{aligned}\right\} \pmod p, \tag{8}$$

$$\left.\begin{aligned} a_{11}x_1+\dots+a_{m1}x_m &\equiv x_{m+1},\\ &\dots\\ a_{1n}x_1+\dots+a_{mn}x_m &\equiv x_{m+n}, \end{aligned}\right\} \pmod p, \tag{9}$$

and let q and Q be the minimal values of $\sum_{i=1}^{n+m} x_i$, where $x_1,\dots,x_{n+m}$ are respectively solutions of systems (8) and (9).

Theorem 2. There exists the following dependence between q and Q:

$$q \geq \frac{cQ^{n+m-1}}{p^{(n-1)(n+m)}\ln p^{(n+m)(n+m-1)}}, \qquad Q \geq \frac{c_1 q^{n+m-1}}{p^{(m-1)(n+m)}\ln p^{(n+m)(n+m-1)}}.$$

Analogous cubic transference theorems are also valid for a system of comparisons.

Using the results of the transference theorems for comparisons, one can prove the following theorem, which somewhat extends the classical transference theorems indicated in the introduction.

Theorem 3. *Let the inequalities*

$$\max\langle M_i(u_1,\dots,u_m)\rangle \leqslant D, \quad |u_j| \leqslant Y$$

$$(i=1,2,\dots,n;\ j=1,2,\dots,m)$$

have no solutions except the trivial one. Then the inequality

$$\prod_{i=1}^{n}\overline{x_i}\prod_{j=1}^{m}\langle L_j(x_1,\dots,x_n)\rangle \leqslant C(m,n)\,\frac{(u^m D^n)^{n+m-1}}{\ln^{(n+m)(n+m-1)}\left(\dfrac{XY}{DC}\right)},$$

has no nontrivial solutions, where

$$X=\max_{i=1,\dots,n}|x_i|, \quad C=\min_{j=1,\dots,m}\langle L_j x_1,\dots,x_n\rangle.$$

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Note: Figure translations are in progress. See original paper for figures.