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Abstract

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ON A PROBLEM OF P. S. URYSOHN

(Presented by Academician P. S. Aleksandrov on 13 II 1963)

1. The **relative distance** $\rho^*(x_1, x_2)$ between points x_1, x_2 of a continuum X is defined as the lower bound of the diameters of those of its subcontinua which contain both these points.* The function $\rho^*(x_1, x_2)$ transforms X into a new metric space X^* (generally speaking, without a countable base), which is called the **space of relative distances** of X. This space was studied in detail by P. S. Urysohn ([1]). The problem of P. S. Urysohn that is under consideration here (problem γ) is the following:

Does there exist a continuum whose space of relative distances is zero-dimensional?

It turns out that this problem has a positive solution. Namely:

*It is possible to construct a continuum C such that C^* is a zero-dimensional space of countable weight.***

Below we give the construction of this continuum.

2. **Notation.** The construction of the continuum C is carried out in the Hilbert parallelepiped H^ω, whose points are described by their coordinates:

$$x = (t_1, t_2, \dots, t_k, \dots), \quad 0 \leqslant t_k \leqslant \frac{1}{k}.$$

Along with H^ω, the following of its faces are considered:

a) The n-dimensional parallelepipeds H^n ($n \geqslant 1$), formed by the first n coordinate axes of H^ω: a point $x \in H^\omega$ belongs to H^n if and only if $t_k = 0$ for all $k > n$. Points of H^n are described by a set of their n coordinates:

$$x = (t_1, t_2, \dots, t_n).$$

b) The rectangles H_n^2 ($n \geqslant 1$), formed by the axes t_n and t_{n+1}: a point $x \in H^\omega$ belongs to H_n^2 if and only if $t_k = 0$ for all k different from n

and from $(n+1)$. Points of the rectangle H_n^2 are described by their two coordinates, which, to avoid confusion, are enclosed not in parentheses but in braces:

$$x = \{t_n, t_{n+1}\}.$$

3. **Auxiliary construction.** Here a continuum Φ_n is constructed, lying in the rectangle H_n^2. For its construction, the lower base of this rectangle (i.e. the segment $0 \leqslant t_n \leqslant \frac{1}{n}$, $t_{n+1} = 0$) is divided into n equal parts by the points

$$x_0, x_1, x_2, \dots, x_n,$$

where

$$x_\nu = \left\{\frac{\nu}{n^2}, 0\right\}.$$

On each segment $[x_{\nu-1}, x_\nu]$ $(\nu = 1, 2, \dots, n)$*** such a po–

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* The notion of relative distance was introduced by Mazurkiewicz.
** And, consequently, homeomorphic to some subset of the set of irrational numbers.
*** By $[x, x']$ is denoted the segment joining the points x and x'.

sequence of points

$$\dots x_\nu^{-m}, \dots, x_\nu^{-1}, x_\nu^0, x_\nu^1, \dots, x_\nu^m, \dots,$$

such that $(\lim_{m\to-\infty} x_\nu^m) = x_{\nu-1}$, $(\lim_{m\to\infty} x_\nu^m) = x_\nu$, and, for $m_2 > m_1$, the point $x_\nu^{m_2}$ lies between the points $x_\nu^{m_1}$ and x_ν.

To each x_ν^m there corresponds a point y_ν^m, lying on the upper base H_n^2: if $x_\nu^m = \{t_n^0, 0\}$, then

$$y_\nu^m = \left\{t_n^0, \frac{1}{n+1}\right\}.$$

By definition,

$$Z_\nu = \bigcup_{m=-\infty}^{\infty} \left([x_\nu^{2m}, y_\nu^{2m+1}] \cup [x_\nu^{2m}, y_\nu^{2m-1}]\right);$$

$$\Phi_n = \left[\bigcup_{\nu=1}^{n} Z_\nu\right]^*.$$

It is easy to show that Φ_n is a continuum.

4. The basic construction. Let ψ_n and φ_n be the natural projections of the parallelepiped H^{n+1} onto its faces H^n and H_n^2, respectively: if

$$x = (t_1, t_2, \dots, t_n, t_{n+1})$$

is a point of H^{n+1}, then

$$\psi_n(x) = (t_1, t_2, \dots, t_n),$$

$$\varphi_n(x) = \{t_n, t_{n+1}\}.$$

To an arbitrary compactum C_n lying in H^n there is assigned a compactum $a_n(C_n) \subseteq H^{n+1}$, defined as follows:

$$a_n(C_n) = \psi_n^{-1}(C_n) \cap \varphi_n^{-1}(\Phi_n).$$

It is easy to show that $\pi_n = \psi_n|_{a_n(C_n)}$ is a** nearly contractive*** monotone mapping of $a_n(C_n)$ onto C_n:

$$\pi_n : a_n(C_n) \to C_n.$$

Let now

$$C_1 = H^1, \quad C_2 = a_1(C_1), \dots, \quad C_{n+1} = a_n(C_n), \dots$$

The spaces C_n, together with the projections $\pi_n : C_{n+1} \to C_n$, form a spectrum. By definition,

$$C = \varprojlim \{C_n, \pi_n\}.$$

Since all C_n are compacta, it follows from the connectedness of C_1 and the monotonicity of π_n that C is a continuum.

5. It can be shown that the continuum C is the desired one, i.e. that C^* is a zero-dimensional space of countable weight.

Below are formulated the assertions on which this proof is based.

Definition. Let X be a topological space and $x \in X$. A set $\widetilde{U} \subseteq X$ containing the point x is called a **relative neighborhood** of this point if it is a component of a set open in X.

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* Square brackets denote closure.

** If $f : X \to Y$ and $A \subseteq X$, then $f|A$ denotes the same mapping f, but considered only on A.

*** A mapping $f : X \to Y$, where X and Y are metric spaces, is called **nearly contractive** if the diameter of any $A \subseteq X$ is not less than the diameter of $f(A)$.

The system of relative neighborhoods turns X into a new topological space, denoted below by $\widetilde{X}$ and called the **space of the relative topology** of X.

The interest of this notion is determined by the following theorem:

Theorem 1. *If X is a continuum, then the spaces X^* and $\widetilde{X}$ are homeomorphic.*

It can be proved that if the mapping $\pi : X \to Y$ is continuous, then the mapping induced by it, $\tilde{\pi} : \widetilde{X} \to \widetilde{Y}$, is also continuous. Thus, every spectrum $\{X_n, \pi_n\}$ induces a spectrum $\{\widetilde{X}_n, \tilde{\pi}_n\}$. The limit space of this spectrum will be denoted below by $\dot{X}$ ($\dot{X} = \varprojlim\{\widetilde{X}_n, \tilde{\pi}_n\}$).

Theorem 2. *If the projections π_n are closed, bicompact and monotone, and $X = \varprojlim\{X_n, \pi_n\}$ is regular, then $\widetilde{X}$ and $\dot{X}$ are homeomorphic.*

In view of Theorems 1 and 2, it is enough to show that $\dot{C} = \varprojlim\{\widetilde{C}_n, \tilde{\pi}_n\}$ is a zero-dimensional space of countable weight. This fact follows from the following theorem:

Theorem 3. *Let $\{X_n, \pi_n\}$ be a spectrum in which all X_n are metrizable compacta and the π_n are nearly confluent mappings. If:*

a) *for every $\varepsilon > 0$ there exists an n_0 such that for $n > n_0$ every point $x \in X_n$ is contained in a connected set $F_n \subseteq X_n$ of diameter less than ε, open and at the same time closed in $\widetilde{X}_n$;*

b) *for every n, $w(\widetilde{X}_n) = \aleph_0$,*

*then $w(\dot{X}) = \aleph_0$, ind $\dot{X} = 0$**.

Consideration of the concrete construction of items 3 and 4 makes it possible to establish that the conditions of Theorem 3 are fulfilled in this construction. In checking condition b), the following theorem is used:

Theorem 4. *$w(\widetilde{X}) > w(X)$ if and only if there exists an open set $U \subseteq X$ such that the cardinality of the set of components of U is greater than $w(X)$.*

6. One may assert the existence of such a continuum F_k of arbitrary positive dimension k***, that F_k^* is a zero-dimensional space of countable weight.

As F_k it suffices to take the topological product of k copies of the continuum C constructed above. The required properties of the space F_k^* are ensured by the following theorem, in which $A = \{a\}$ is a set of indices, and the X_a are arbitrary topological spaces.

Theorem 5. *Let* $X = \prod_{a \in A} X_a$, *and* $\widehat{X} = \prod_{a \in A} \widetilde{X}_a$. *If all the* X_a, *except perhaps finitely many, are connected, then the spaces* $\widehat{X}$ *and* $\widetilde{X}$ *are homeomorphic.*

The example of the topological product of countably many simple dyads shows that the connectedness condition in this theorem is essential.

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CITED LITERATURE

1. P. S. **Uryson**, *Works on topology and other areas of mathematics*, **2**, 1951, p. 517.

* $w(X)$ denotes the weight of the space X.
** ind is the small inductive dimension.
*** And even infinite-dimensional.

Note: Figure translations are in progress. See original paper for figures.