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Abstract

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ON THE INFLUENCE OF FRICTION ON OCEAN CURRENTS

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Let us consider the problem of stationary currents in the ocean caused by wind. To analyze the features of complete flows, we consider the following equation for the stream function ψ:

$$-\gamma\Delta\Delta\psi + \varepsilon(\psi_x\Delta\psi_y - \psi_y\Delta\psi_x) + \psi_x + g(y) = 0. \tag{1}$$

Equation (1) is written in dimensionless form; the x-axis is directed eastward and the y-axis northward. It is assumed that the meridional component of the wind stress is equal to zero, while the zonal component τ depends only on y; $g(y) = \tau'(y)$.

We shall regard the positive parameters ε and γ as constants. The parameter ε characterizes the influence of vorticity advection, while γ characterizes the influence of vorticity dissipation associated with horizontal turbulent exchange. The parameters ε and γ are small quantities.

The model described was proposed by Carrier and Robinson in work (^{1}). In that paper, a solution of equation (1) was considered with the boundary conditions

$$\psi\big|_\Gamma = 0; \tag{2}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_\Gamma = 0 \tag{3}$$

in a closed domain D with boundary Γ, under the condition that $\gamma \ll \varepsilon \ll 1$. It was assumed here that the solution of problem (1), (2), (3) is close to some solution of the equation

$$\varepsilon(\psi_x\Delta\psi_y - \psi_y\Delta\psi_x) + \psi_x + g(y) = 0 \tag{4}$$

everywhere in the domain D, with the possible exception of boundary layers and certain narrow internal regions.

A solution of problem (4), (2) was constructed in (^{1}), whose characteristic feature is the separation of the boundary layer in ε from the western wall at the point where $g'(y) = 0$, and the presence of a narrow inertial jet inside the domain. The solution of this problem was applied to explain the separation of the Gulf Stream and the Kuroshio from the coast.

In work (^{1}) it was indicated that the solution of problem (4), (2) is not unique. It must be noted that none of these solutions can be supplemented by boundary layers in γ to give a solution of problem (1), (2), (3).

Apparently this means that the influence of friction for $\gamma \ll \varepsilon$ is not always confined only to narrow regions, and therefore it is not natural to expect that, for small γ, the solution of problem (1), (2), (3) is close to some solution of equation (4) everywhere in the domain D outside sufficiently narrow regions.* We shall construct a solution of problem (1), (2), (3) illustrating this assertion.

We shall consider such a relation between the parameters γ and ε for which the dissipative terms and advection have the same order at the western boundary. To this end we put $\gamma = a\varepsilon^{3/2}$, where $a > 0$ and does not depend on ε. In what follows, by increasing the parameter a, we obtain Munk' s model (^{3}); decreasing the parameter a leads to the assumption $\gamma \ll \varepsilon$ of work (^{1}).

* A similar phenomenon arises in diffuser flow for large Reynolds numbers (see (^{2})).

As the domain D, take the square $0 \leqslant x \leqslant 1;\ 0 \leqslant y \leqslant 1$, and consider problem (1), (2), (3) with a linear function $g(y)$: $g(y) = by + c$. For fixed a we obtain a problem with one small parameter ε. We shall seek the solution of this problem in the form

$$\psi = g(y)F(x) \tag{5}$$

everywhere in D, for the time being not paying attention to conditions (2), (3) at the boundaries $y = 0$ and $y = 1$. Substituting (5) into (1), (2), (3), we have:

$$-a\varepsilon^{3/2}F^{(\mathrm{IV})} + \varepsilon b(F'F'' - FF''') + F' + 1 = 0 \tag{6}$$

with the conditions

$$F(0) = F'(0) = 0; \qquad F(1) = F'(1) = 0. \tag{7}$$

Let us construct the solution of problem (6), (7). We apply the usual method of expanding the solution of a boundary-value problem with a small parameter multiplying the highest derivatives into an asymptotic series in powers of ε (see (^{4,5})). Put:

$$F(x) = F_0(x) + \sqrt{\varepsilon}\,F_1(x) + \dots + w_0\left(\frac{x}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\,w_1\left(\frac{x}{\sqrt{\varepsilon}}\right) + \dots$$

$$\dots + v_0\left(\frac{1-x}{\sqrt{\varepsilon}}\right) + \sqrt{\varepsilon}\,v_1\left(\frac{1-x}{\sqrt{\varepsilon}}\right) + \dots \tag{8}$$

The functions $F_k(x)$ are the coefficients of the expansion of the solution in a series inside the domain D. The functions w_k and v_k are substantially different from zero only inside small neighborhoods of the points $x = 0$ and $x = 1$ and compensate the discrepancies in satisfying the boundary conditions (7) by the functions $F_k(x)$.

Substituting (8) into (6), (7), we obtain equations for $F_k(x)$, $w_k(\xi)$, and $v_k(\xi)$

$$\left(\xi = \frac{x}{\sqrt{\varepsilon}} \geqslant 0;\ \xi = \frac{1-x}{\sqrt{\varepsilon}} \geqslant 0\right):$$

$$F_0' + 1 = 0, \tag{9}$$

$$\dots$$

$$-aw_0^{(\mathrm{IV})} + b\left[w_0' w_0'' - (w_0 + F_0(0))w_0'''\right] + w_0' = 0, \tag{10}$$

$$\dots$$

$$-av_0^{(\mathrm{IV})} - b\left[v_0' v_0'' - (v_0 + F_0(1))v_0'''\right] - v_0' = 0; \tag{11}$$

$$\dots$$

and the boundary conditions ($k = 0, 1, 2, \dots$):

$$w_k(0) + F_k(0) = 0; \qquad w_0'(0) = 0; \qquad w_{k+1}'(0) + F_k'(0) = 0; \tag{12}$$

$$v_k(0) + F_k(1) = 0, \qquad v_0'(0) = 0; \qquad -v_{k+1}'(0) + F_k'(1) = 0. \tag{13}$$

In the sense of expansion (8), the functions $w_k(\xi)$ and $v_k(\xi)$ must decay exponentially at infinity. It is natural to assume that the behavior of such solutions of equations (10) and (11) for large ξ is determined by the linear terms of these equations. After one integration and linearization, we obtain:

Figure 1 schematic

Figure 1: Figure 1 schematic

$$-aw_0''' - bF_0(0)w_0'' + w_0 = 0; \qquad \xi \geqslant 0; \tag{10'}$$

$$-av_0''' + bF_0(1)v_0'' - v_0 = 0; \qquad \xi \geqslant 0. \tag{11'}$$

Since $a > 0$, equation $(10')$ has two linearly independent solutions that decay at infinity, while equation $(11')$ has one such solution. Hence one may conclude that equation (10) has a two-parameter family of exponentially decaying solutions, whereas equation (11) has only a one-parameter family of such solutions.

By virtue of the condition $v_0'(0) = 0$ we obtain $v_0(\xi) \equiv 0$ and, consequently, $F_0(1) = 0$. Then, by (9), $F_0(x) = 1 - x$. Next, by the usual method, we find all the functions $F_k(x)$ and $v_k(\xi)$. Conditions (12) determine the functions $w_k(\xi)$.

Let us consider in more detail the decreasing solutions of equation $(10')$ for different b. They have the form $c_1 \exp \lambda_1 \xi + c_2 \exp \lambda_2 \xi$, where λ_1 and λ_2 are the roots of the polynomial $-a\lambda^3 - b\lambda^2 + 1 = 0$, for which $\operatorname{Re} \lambda < 0$.

Consider two cases:

1) $b > 0$; for small values of a we obtain that

$$\lambda_1 = -\frac{1}{\sqrt{b}} + O(a),$$

$$\lambda_2 = -\frac{b}{a} + O(a),$$

and thus the boundary layer in ε is, as it were, split into two boundary layers of thicknesses of orders $\sqrt{\varepsilon}$ and $\sqrt{\varepsilon}$, respectively.

Fig. 1. Diagram of the streamlines of the depth-averaged wind-driven current. The function $g(y)$ is shown on the right. At the top and bottom are schematic graphs of the functions $\hat{F}(x)$ and $\check{F}(x)$. The orders of the thicknesses of the corresponding boundary layers are indicated in the diagram.

2) $b < 0$; for small values of a we obtain that

$$\lambda_1 = \frac{i}{\sqrt{-b}} - \frac{a}{2b^2} + O(a^2),$$

$$\lambda_2 = -\frac{i}{\sqrt{-b}} - \frac{a}{2b^2} + O(a^2).$$

Consequently, the boundary layer in ε in this case is stretched and has thickness of order $\sqrt{\varepsilon}/a$, while the functions $w_k(x/\sqrt{\varepsilon})$ oscillate, decreasing slowly.

It is natural to expect that the boundary layers determined by the nonlinear equations (10), (11) have the corresponding characteristic features. Thus, the sign of the quantity $b = g'(y)$ substantially affects the behavior of the solution of problem (6), (7). We note that also in work (^{1}), when considering equation (4), the sign of $g'(y)$ played an essential role.

Let us now construct the solution of problem (1), (2), (3) in the square D with the piecewise-linear function $g(y)$

$$g(y) = \begin{cases} y, & \text{for } y \leq \frac{1}{2}, \\ 1 - y, & \text{for } y \geq \frac{1}{2}. \end{cases}$$

Outside the boundary layers at the boundaries $y = 0$ and $y = 1$, we represent the solution in the form (5) with different functions $\check{F}(x)$ and $\hat{F}(x)$ for the lower and upper halves of the square, respectively. In determining the function $\check{F}(x)$, the derivative $g'(y) > 0$, while in determining $\hat{F}(x)$ the derivative $g'(y) < 0$, and we obtain the patterns analyzed above.

It is not difficult to show that, by constructing boundary layers at the boundaries $y = 0$ and $y = 1$, one can achieve satisfaction of conditions (2), (3) on the whole boundary Γ.

It is easy to see that the constructed solution, for large values of the parameter a and small ε, continuously passes into the solution of the linear problem, obtained —

obtained by Munk in (^{3}). For small but finite values of a, the isolines of the function ψ are shown schematically in Fig. 1.

For very small a, problem (1), (2), (3) is equivalent to the problem considered in (^{1}). In this case the solution in the lower half of the square ($g' > 0$) is close to the solution of equation (4). In the upper part of the domain ($g' < 0$) the boundary layer at the western wall spreads out as $a \to 0$, and the structure of the function ψ may then be far from the structure of the solutions of equation (4).

The solution of the problem indicated above has a discontinuity in the boundary layer near the boundary $x = 0$ at $y = 1/2$, as well as a discontinuity along the line $y = 1/2$ in terms of the second and higher orders. We believe that the cause of this is the discontinuity of the derivative g' at $y = 1/2$. If, however, one takes as $g(y)$ a smooth positive function with one maximum, equal to zero at $y = 0$ and $y = 1$, then, apparently, there will be no internal discontinuities in the solution of problem (1), (2), (3), while the character of the boundary layers in the upper and lower halves of the square will be preserved.

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Note: Figure translations are in progress. See original paper for figures.